# Liquidation as a Market: Competition, Congestion, and Equilibrium in DeFi Lending

Jad Soucar

Daniel J. Epstein Department of Industrial & Systems Engineering
University of Southern California, Los Angeles, U.S.A
`soucar@usc.edu`

**Abstract**

Over-collateralized lending protocols rely on liquidation incentives to maintain solvency, yet key parameters such as liquidation bounties, collateral factors, and liquidation thresholds are often fixed or adjusted only through slow governance processes. This paper develops a game-theoretic framework for dynamically adjusting these parameters in decentralized lending markets with competing protocols. We model borrowers, lenders, and liquidators as populations that choose among protocol–collateral pairs as a population game, while the upper-level interaction between protocols is modeled as an equilibrium problem with equilibrium constraints. We introduce a variationally stationary Nash concept that reflects the fact that protocols directly control only their own parameters while the market equilibrium state is induced indirectly through participant behavior. To compute such points we reformulate the EPEC as a linkage problem, and propose a homotopy progressive decoupling algorithm. Under elicitable maximal monotonicity and graphical convergence assumptions, cluster points of the algorithm induce variationally stationary Nash points of the original EPEC. The framework provides a means protocols to dynamically update parameters in decentralized lending markets. We conclude with numerical experiments highlighting how protocols who use such an algorithm can improve conditions for borrowers while maintaining systemic stability of the multi-protocol lending market.

**Keywords:** Decentralized lending markets, game theory, progressive decoupling

**JEL Classification:** G23; G28; C62; C61; C72; D47

## 1 Introduction

Over-collateralized lending protocols (OLPs), have become an increasingly important tool used by cryptocurrency owners who wish to operationalize their crypto assets. Simply put, overcollateralized lending is the process of lending capital to a borrower who has posted collateral of higher value than the loan itself. This is not a new idea. Take for example an individual who wants access to capital. Instead of selling their illiquid assets such as property, they instead post these assets as collateral and borrow against them.

To understand how such a process can be decentralized and implemented on a blockchain, consider Alice and Bob. Assume Bob wishes to loan 100 units of crypto asset A, which has a fixed nominal value of one dollar per unit. Also assume that Alice owns 100 units of crypto asset B, which has a fluctuating nominal value of one dollar per unit. Like the hypothetical property owner, Alice wishes to obtain liquid capital without selling her existing crypto assets since her asset might appreciate in value. To do this Alice posts collateral of 100 units of asset B, then takes out a loan of 75 units of asset A from Bob. In this way Alice's loan remains overcollateralized with a debt to collateral ratio of $0.75$. Often times a decentralized market will enforce that the debt to collateral ratio remains bellow some constant refered to as a liquidation threshold. Assume that Alice takes asset A, sells it, invests the proceeds, and nets a $g\%$ gain. In the best case scenario asset B, which serves as Alice's collateral, increased in value to two dollars per unit. In which case Alice, sells her investment, repurchases 75 units of asset A at one dollar per unit, repays her loan of 75 units of asset A plus interest of rate $r$, then retrieves her collateral of 100 units of asset B. In total Alice has a net gain of $75(g - r) + 100$ and the lender has a net gain of $75r$.

Blockchain systems lack legal recourse, so when the collateral looses instead of gains value, the protocol must ensure that loans remain solvent through purely algorithmic means. For example if the value of Alice's debt to collateral ratio exceeds a specified threshold, then Bob should be worried that his loan may no longer be paid back in full. In this case Alice's collateral is eligible for liquidation to pay back the lender. Otherwise Alice can settle her debt whenever she chooses. Importantly the process of liquidating collateral is not automatic and is not driven by a central authority. Instead a liquidator is allowed to repay a portion of the debt to the lender, and in return the liquidator receives a value-equivalent portion of the collateral plus some reward called a liquidation bounty. Without liquidators a lending protocol can accumulate "bad" debt resulting in some borrowers being unable to repay the lender. As a result overcollateralized lending protocols rely on carefully engineered interactions between borrowers, lenders, and liquidators which are designed to incentivize borrowers to borrow, lenders to lend, and liquidators to liquidate bad and under-collateralized loans. These OLPs often compete with each other to attract borrowers, lenders, and liquidators to their protocol.

## 1.1 Research Question

In practice, many of the protocol's operational parameters such as liquidation bounties are fixed. In the rare occasion when the parameters are changed, protocols call upon the market participants to vote on such changes. These "governance votes" can take anywhere from one day to weeks, which means that protocols are unable to adjust to market conditions in real time. For example liquidation bounties are often fixed at approximately $5\%$ of the liquidated collateral value. Such rewards ensure that liquidators are incentivized to monitor loans and clear "bad" debt from the protocol. However liquidation bounties create potentially unnecessary costs to borrowers since the liquidation bounty is deducted directly from their collateral during liquidation events. This raises the question of whether large fixed liquidation bounties and other parameters are required to incentivize sufficient participation of liquidators,lenders, and borrowers in the market? The primary research question we wish to address is then the following.

**Research Question:** Can an algorithmic controller be developed that allows a protocol to dynamically adjust liquidation bounties and other protocol parameters in order to reduce effective borrower fees and while still retaining sufficient market participation of borrowers, lenders, and liquidators.

To answer this questions we formulate a bi-level game to model competition among protocols for market participation. By finding the equilibrium of such a game we determine whether or not a smaller liquidation bounty is feasible while preserving the stability of the lending system. Additionally, by analyzing the equilibrium of this game, we can characterize how market participants respond to protocol parameters. We can then use this relationship to design feedback rules that adjusts previously fixed or inelastic protocol parameters.

## 1.2 Prior Work

A growing body of research studies have aimed to understand how protocol parameters can be used to balance the participation of borrowers, lenders, and liquidators in OLPs. Specifically, recent work has focused on adaptive mechanisms for adjusting interest rates. For example, Bastankhah et al. (2024a) proposes a two-timescale controller that dynamically adjusts lending interest rates in order to stabilize utilization and liquidity conditions. Similarly, Bastankhah et al. (2024b) develops a learning-based mechanism that adapts lending rates based on observed supply and demand in DeFi markets.

However this line of work has largely focused on interest rate adjusting algorithms whose primary effect is to influence the participation of borrowers and lenders. Other important protocol parameters such as the liquidation threshold and liquidation bounty interact primarily with liquidators rather than borrowers and lenders. Despite the central role the liquidators play in maintaining protocol solvency, algorithmic means of controlling these liquidator-focused parameters has been largely ignored. Such algorithmic controllers are of particular interest since liquidators serve as the primary enforcement mechanisms of the OLP. Without a sufficient population of liquidators under-collateralized loans may persist, potentially leading to default and reputational damage to the protocol. Additionally, most existing work studies individual lending protocols in isolation. In reality DeFi lending markets consist of multiple competing protocols that simultaneously attempt to attract borrowers, lenders, and liquidators. For example, a protocol that sets overly conservative parameters may lose market participants to other protocols offering more attractive borrowing or lending conditions. On the other hand, a protocol that sets overly aggressive parameters risks alienating market participants or accumulating bad debt.

There has been growing interest in empirically understanding the role of such parameters on OLPs. For example, Qin et al. (2021) provides one of the first systematic empirical studies of liquidation events across major DeFi lending protocols and shows that liquidation incentives significantly influence both market outcomes and borrower losses. Related work such as Perez et al. (2021) studies the sensitivity of DeFi lending systems to liquidation parameters and shows that relatively small changes in these parameters can substantially affect liquidation efficiency and protocol stability. Taken together, these studies suggest that liquidation incentives play an important role in improving the solvency of OLPs. Additionally they suggest that dynamically adjusting such parameters may improve the efficiency of lending systems, just as algorithmic control of interest rates has been shown improved market stability. Despite these recent results, to our knowledge, no algorithmic controller of liquidator-focused parameters exists in the literature that incorporates inter-protocol competition.

## 2 Proposed Game-Theoretic Model

The system of incentives that make up overcollateralized lending markets can be formulated as a game, where protocols adjust their operational parameters to (a) attract borrowers who seek low effective interest rates, (b) attract lenders who seek high effective interest rates, and (c) attract liquidators to quickly clear "bad" debt from the protocol. In response to the rates set by each protocol, the borrowers, lenders, and liquidators flow from one protocol to another. Such a game can be modeled as an equilibrium problem with equilibrium constraints, where the upper level game is a sequential game played between protocols and the lower level game is a population game between three populations; the borrowers, lenders, and liquidators. In order to formulate such a game we define the environment, the protocol actions, the lower level game between market participants, and the full formulation of the equilibrium problem with equilibrium constraints.

### 2.1 Environment

Let there be $N$ different lending protocols which are facilitating the lending interactions and that there are $K$ different assets that the $N$ lending pools can choose to accept as collateral. Each collateral asset $\{p_i\}_{k=1}^{K}$ is assumed to be a geometric Brownian motion with drift such that

$$dp_k(t) = p_k(t)(\mu_k dt + \sigma_k dW_t^k), \tag{1}$$

for some asset specific drift $\mu_k$ and volatility $\sigma_k$. For the sake of simplicity we assume the following.

**Assumption 2.1** (Collateral and Borrowed Assets & Access)**.** *We assume that there is only one asset being borrowed and that asset is pegged to a stable currency like the U.S Dollar. We denote the price of the stable asset as* $p_0(t) = 1$ *for all* $t$*. We also assume that* $p_k(t)$ *are expressed in a common numeraire for all* $k = 0, \dots, K$*. The lenders, borrowers, liquidators, and protocols are assumed to have access to an oracle model which provides agents with the current prices of all assets at any time* $t$*.*

Assumption 2.1 is reasonable since in practice the dominant use case of overcollateralized lending is borrowing a dollar-denominated stable coin against volatile crypto collateral (Chiu and Danisman, 2026). We also impose that each protocol $i$ imposes a limit on how much can be loaned. We refer to such a limit as the Total-Loan-Value or $\text{TLV}_i$ for $i = 1 \dots, N$.

#### 2.1.1 Interest Rates

We assume that for a protocol $i$, borrowers pay an interest rate $r_i$. Its important to note that the lenders only receive interest payments on the assets that have been borrowed. In other words if the total value loaned and borrowed on chain $i$ is $L_i$ and $B_i$ respectively then the total utilization ration is $B_i/L_i = U_i$. The interest payment are then calculated as $(r_i - \rho_i)\ell_i \cdot U_i$, where $\ell_i$ is the total amount loaned by specific lender on protocol $i$ and $\rho_i$ is a protocol fee. Typically a protocol takes $\rho_i\%$ of the interest rate payments and either treats it as income or places it into a reserve to cover bad-debt events. In many overcolateralized lending protocols $r_i$ is a function of the total utilization rate $U_i$. For example the interest rate formula used by the Aave (2020) protocol is

$$\tilde{r}_i(U_i) = \begin{cases} r_i^0 + \frac{U_i}{U_i^{opt}} r_i^1 & u_i \leq u_i^{opt} \\ r_i^0 + \frac{U_i - U_i^{opt}}{1 - U_i^{opt}} r_i^2 & u_i > u_i^{opt} \end{cases},$$

where $r_i^1 \ll r_i^2$. Such a structure attracts lender to the protocol when the utilization rate is above some protocol specific optimal utilization rate $U_i^{opt}$. On the other hand, when the utilization rate is bellow $U_i^{opt}$ borrowers are attracted to the protocol due to low interest rates. For the purpose of this work we use the smoothened form with small $\delta > 0$,

$$r_i(U_i) = r_i^0 + \frac{r_i^1}{U_i^{opt}} U_i + \left( \frac{r_i^2}{1 - U_i^{opt}} - \frac{r_i^1}{U_i^{opt}} \right) \frac{1}{2} \left( U - U_i^{opt} + \sqrt{(U - U_i^{opt})^2 + \delta^2} - \frac{\delta}{2} \right). \tag{2}$$

### 2.1.2 Gas Fees & Tips

In their simplest form block-chains are distributed ledger systems that record market participants' transactions. In practice, every time a transaction such as a borrowing, lending, or liquidating action is taken, the corresponding transaction is stored in a publicly observable list of pending transactions referred to as a mempool. In order for a transaction to enter the mempool the initiator of that transaction is required to pay a "gas fee", which roughly corresponds to the resource consumption of the transaction. For the purpose of the proposed model we fix the gas fee and other interest rate parameters, as they are not the primary parameters of interest.

**Assumption 2.2** (Fixed Quantities). *We assume that that the gas fee is some constant $g$, the interest rate parameters $\{(r_i^0, r_i^1, r_i^2)\}_{i=1}^N$ are fixed, and the rate $r_f$ (which denotes the average yield of a risk free treasury bond) is kept constant across all protocols.*

These transaction are not immediately recorded. Instead specialized actors referred to as block-builder assemble candidate blocks by selecting and ordering transactions to maximize total extractable value (MEV). Block builders then compete to supply their blocks to a block validator, who selects the most profitable block for inclusion in the block chain, reaps the profit, and provides the block-builder with a financial kick-back. Typically a maximum of 1 block gets built during a "block-time". For example in the Ethereum block-chain, one block gets built every 12 seconds (ethereum.org, 2026). As a result market participants who wish to complete a transaction quickly can send a "tip" along with their transaction. If the tip is sufficiently large, then the block-builder will have an incentive to include the transaction in their block in order to increase the likelihood that they are selected by the block validators. This system creates a sub economy within the block-chain where (a) searchers aim to build bundles with large tip in order to get a larger kick-back from the block builder (b) block builders select the searcher who's bundle offers the highest tip (c) market participants who wish to execute a transaction quickly will pay a gas fee and offer a high tip to earn inclusion in the winning block that gets built.

## 2.2 Protocol Actions

In the upper level game each of the $N$ protocols adjust their operational parameters in order to attract market participants. Each protocol can choose to adjust a vector of parameters

$$a_{ik} = \begin{pmatrix} \lambda_{ik} & \theta_{ik} & \gamma_{ik} \end{pmatrix}, \quad a_i = \begin{pmatrix} a_{i,1} & \dots & a_{ik} \end{pmatrix} \in \mathcal{A}_i, \tag{3}$$

where $\lambda_{ik}$ is the collateral factor for each asset type $k$. For example if $\lambda_{ik}$, corresponding to the collateral factor on the $i^{th}$ protocol for collateral asset type $k$, equals $0.75$ then a borrower can take a loan with value up to $75\%$ of their posted collateral. This ensures that a loan is overcollateralized, which means that there is low risk of the lender losing the value they've loaned. Additionally each protocol must define a liquidation threshold $\theta_{ik}$, which defines when a liquidator is allowed to liquidate the collateral of type $k$ on protocol $i$. For example if $\theta_{ik} = 0.90$ then the collateral can be liquidated when a borrower's debt exceeds or reaches $90\%$ of the collateral's value in order to pay back part of the borrower's debt. The liquidator is rewarded for liquidating the collateral and paying back the borrowers debt with a liquidation bounty $\gamma_{ik}$, which corresponds to the percentage of the borrower's liquidated collateral that a liquidator gets paid. When $\gamma_{ik}$ is sufficiently large, this system incentivizes prompt removal of "bad" debt from the lending pool. In this setting the feasible action set $\mathcal{A}_i = ([\underline{\lambda}, 1] \times [\underline{\theta}, 1] \times [0,1])^{3K}$ and $a = (a_1, \dots, a_N) \in \mathcal{A} = [0,1]^{3KN}$. Note that both $\lambda_{ik}$ and $\theta_{ik}$ are bounded away from 0 with lower bounds $\underline{\lambda} > 0$ and $\underline{\theta} > 0$. The lower bounds on rule out degenerate protocol parameters. For instance, a collateral factor $\lambda_{ik} = 0$ prevents borrowers from borrowing against any amount of collateral, while a liquidation threshold $\theta_{ik} = 0$ would make any positive debt-to-collateral ratio immediately liquidatable. The bounds restrict attention to economically meaningful lending markets.

### 2.2.1 Protocol Driven Liquidation Mechanics

We emphasize that the protocol's actions impact liquidations. Specifically, liquidations in OLPs occur when the debt to collateral value ratio exceeds $\theta_{ik}$ for a particular asset type $k$ on protocol $i$. Liquidation typically unfold over two steps. First the liquidator pays back the loan then the collateral is liquidated to pay back the liquidator plus some liquidation bounty. In this way liquidation can be viewed as a exchange of the loan asset for the collateral asset. Note the the full collateral is not liquidated. In order to ensure that any borrower who has breached the liquidation threshold is reverted to a state where their debt to collateral ratio is bellow $\theta_{i,k}$, the minimum liquidation is given by lemma 2.3 and was first presented by Bastankhah et al. (2024a).

**Lemma 2.3** (Minimum Liquidation)**.** *Assuming that liquidation are completed instantaneously with no price slippage. For a fixed borrower with debt $d_{ik}(t)$ on protocol $i$ whose posted $c(t)$ units of collateral $k$ with price $p_k(t)$. Assume that $d_{ik}(t)/c(t)p_k(t) > \theta_{ik}$. Then the minimum value $\ell$ that can be liquidated subject* *to the liquidation bounty* $\gamma_{ik}$ *is*

$$\ell_{\min}(t) = \max\left\{0, \frac{d_{ik}(t) - \theta_{i,k}c(t)p_k(t)}{(1-\theta_{ik}(1+\gamma_{ik}))}\right\}$$

Additionally many protocols, require a minimum liquidation of size $\ell$ but allow maximum collateral of $c(t)\chi(h(t))$. Where $\chi(h(t)) \in [0,1]$ is the close factor function, which determines the percentage of the maximum collateral that can be liquidated once the liquidation threshold has been reached. The input of the close factor function is typically the health factor $h(t)$ defined as the $\theta c(t)p(t)/d(t)$ where $d(t)$ is the value of debt of the borrower and $c(t)$ is units of collateral posted. In many lending protocols such as Aave (2020) the close factor function is defined as

$$\chi(h(t)) = \begin{cases} 0 & h(t) > 1 \\ 0.5 & h(t) \in (0.95, 1] \\ 1 & h(t) \leq 0.95 \end{cases}, \tag{4}$$

which we will use for the remainder of this work. As a result, the liquidator can liquidate anywhere from $\ell_{\min}(t)(1+\gamma)$ to $c(t)\chi(h(t))(1+\gamma)$ when the liquidation threshold is met or breached. Observe that the liquidator also takes an additional $\%\gamma$ of the collateral as their bounty.

How an individual liquidator liquidates a bad loan is almost entirely defined by the lending protocol. However liquidators often compete for the right to liquidate the loan and reap the liquidation bounty. Recall from section 2.1.2 that liquidations can be treated as a type of transaction similar to buying or selling an asset on the block chain. As such liquidators must pay the gas fee $g$ to submit their liquidation request. Note that typically liquidators send their requests directly to block builders, via a "flashbot" to avoid their request being publicly observed in the chain's mempool (Li et al., 2023). In order to incentivize the block-creators and validators to select their liquidation request over other competing liquidators, a liquidator will often submit a tip along with their transaction. At the end of the block-chain's block-time a liquidator with the highest tip is typically chosen for inclusion in the new block. We propose that liquidator competition can be represented as a timed sealed bid first price auction since liquidators submit their tip as a closed bid to block creators, who then select the highest value transaction to include in a new block at the end of the block-time. To model the sealed bid first price auction we make a few assumptions then show proposition (2.6), which follows algebraically from standard results in auction theory (Krishna, 2010).

**Assumption 2.4** (Population Counts)**.** *We denote the the number of individual lenders,borrowers, and liquidators in the system as* $N^L$,$N^B$, *and* $N^Q$ *respectively. The number of participants of any population* $p \in \{B, L, Q\}$ *monitoring* $s \in S_p$ *is denoted as* $N^p_s$.

**Assumption 2.5** (Liquidator Auction)**.** *All liquidators will always request to liquidate the maximum amount available, then the bounty received by the winning liquidator is given as* $W_{ik} = p_k(t)c(t)\chi(h_{ik}(t))\gamma_{ik} - g$, *where* $c(t)$ *is the units of collateral posted,* $d_{ik}(t)$ *is the units of asset borrowed (with nominal value of 1 per unit), and* $h_{ik}(t) = \theta_{ik}c(t)p_k(t)/d_{ik}(t)$.

**Proposition 2.6** (Winning Liquidation)**.** *Assuming assumption 2.5 and that each bidder values the opportunity to liquidate as* $\mathrm{Uniform}([0, W_{ik}])$*, then the winning liquidator will win a net reward of*

$$R_{ik}(N_{ik}^{Q}) = \frac{1}{N_{ik}^{Q}+1}(p_k(t)c(t)\chi(h_{ik}(t))\gamma_{ik} - g).$$

## 2.3 Lower Level Game

In the upper level game each of the $N$ protocols adjust their operational parameters in order to attract market participants. Each protocol can choose to adjust a vector of parameters $a_i \in \mathcal{A}^K$. Once each protocol chooses an action $a_i$, we assume that the market participants engage in a population-type game in which they flow from one protocol to another. Specifically, the lower level market participants will choose from a set of discrete actions until they've reached an equilibrium. The borrowers, lenders, and liquidators choose actions from the following discrete sets respectively;

$$\begin{aligned} S_B &= \{(i,k) : i = 1,\ldots,N \quad \text{and} \quad k = 1,\ldots K\} \\ S_L &= \{i : i = 1,\ldots,N\} \\ S_Q &= \{(i,k) : i = 1,\ldots,N, \quad \text{and} \quad k = 1,\ldots K\}. \end{aligned}$$

In other words, the borrowers will choose which protocol $i$ they'd like to borrow from and which collateral they'd like to deposit on that protocol. The lenders choose which protocol they'd like to deposit their stable asset into. Recall, that we assume that only one asset can be borrowed and so the lenders' choice reduces to choosing a protocol. Finally the liquidators face the same choice as the borrowers. Namely they choose which protocol they'd like to monitor for "bad" debt and which type of collateral they are willing to liquidate. Given fixed protocol parameters, $\{a_i\}_{i=1}^{N}$, the lower level market participants eventually reach an equilibrium distribution. We denote the population of borrowers as $x^B$, the population of lenders as $x^L$, and population of liquidators as $x^Q$. We denote the full distribution of populations as $x = (x^B, x^L, x^Q)$. The equilibrium distribution of each lower level market participant must satisfy

$$x^{B*}, x^{Q*} \in \{x : \sum_{(i,k)\in S_B} x_{ik} = 1, x \geq 0\}, \quad x^{L*} \in \{x : \sum_{i \in S_L} x_i = 1, x \geq 0\}.$$

We assume that each population of market participants aims to minimize some cost function which we describe in the following sections. We begin however with some assumptions.

**Assumption 2.7** (Average Quantities)**.** *The average nominal amount loaned per lender on protocol* $i$ *is* $\eta_i^L$*. The average nominal amount posted as collateral of type* $k$ *on protocol* $i$ *is denoted by* $\eta_{ik}^C$*. The average amount borrowed per borrower on protocol* $i$ *using collateral type* $k$ *is given by* $\eta_{ik}^B = \eta_{ik}^C \phi_{ik}$*, where* $\phi_{ik} \leq \lambda_{ik}$*. In other words,* $\phi_{ik} \leq \lambda_{ik}$ *represents the initial debt-to-collateral ratio chosen by borrowers using collateral* $k$ *on protocol* $i$*, which must not exceed the protocol's collateral factor* $\lambda_{ik}$*.*

In practice, the costs faced by borrowers, lenders, and liquidators evolve over time since the value of collateral assets fluctuates continuously according to the stochastic price processes $\{p_k(t)\}_{k=1}^{K}$. As a result, the profitability or risk associated with participating in a particular protocol changes dynamically. To capture this effect we model the costs as being accumulated over a random time horizon.

**Assumption 2.8** (Time Accumulation)**.** *The costs $c^B_{ik}$, $c^Q_{ik}$ and $c^L_i$, as being accumulated over a random time horizon of time $\hat{\tau}_i = \min\{\tau_i, \tau^L_i\}$. We let $\omega_i > 0$ and $\tau_i \sim \mathrm{Exp}(\omega_i)$ which may vary by protocol and $\tau^L_i$ denote the first time that the loan is liquidated*

This reflects the idea that market participants periodically re-balance their positions. However, recall that a loan position may terminate prematurely if it becomes under-collateralized and is liquidated. In this case the borrower or lender may choose to move their capital elsewhere to avoid further liquidations.

### 2.3.1 Borrowers

The borrowers aim to minimize $c^B_{ik}(a, x)$ which is a function of the population distribution $x$ and action profile $a$. It depends on (a) the accessibility of capital, (b) the cost of borrowing using collateral $k$ on protocol $i$, (c) the expected loss due to liquidation of their collateral. We begin with the borrower's accessibility to capital $A^B_{ik}(x)$ which depends on the amount of lender deposits and the protocol specific total loan value cap $TLV_i$. We model this term as

$$A^B_{ik}(x) = \begin{cases} 0 & U_i(x) \le 1, \quad \sum_{k=1}^K x^B_{ik}\eta^B_{ik} \le TLV_i \\ \infty & \text{otherwise,} \end{cases}$$

where $U_i$ is the utilization rate on a protocol $i$ defined as

$$U_i(x) = \frac{N^B}{N^L} \sum_{k=1}^{K} \frac{\eta^C_{ik}\phi_{ik}\, x^B_{ik}}{\eta^L_i\,(x^L_i + \epsilon_i)},$$

where $\epsilon_i$ is small and greater than 0 for all $i = 1, \dots, N$. The term $\epsilon_i$ serves the purpose of preventing discontinuities in $U_i(x)$ and can be interpreted as a protocol always having some small amount of seed liquidity, which is typical on many protocols. In this case capital becoming inaccessible, i.e. more borrowing then lending, is modeled as borrower cost going to $\infty$. The interest rate cost accumulated over the time horizon specified in assumption (2.8) is modeled as $(r_i - r_f)U_i(x)\eta^L_{ik}\,\hat{\tau}_i$, where $r_f$ is the risk free rate. We subtract the risk free rate since borrowers can always leverage their borrowed capital to reap at least and $r_f$ return. Finally, the cost to the borrower depends on amount they loose in the case of a liquidation event. This quantity depends on whether there are liquidators monitoring the protocol, the liquidation bounty, and the amount of collateral liquidated within the time period $\hat{\tau}_i$, which we refer to as $H(\hat{\tau}_i)$. As such we model the loss to liquidation as $\iota_{ik}(x^Q_{ik})\mathbb{E}[H(\hat{\tau}_i)|p_k(t_0)]$, where $H(\hat{\tau}_i)$ is given by proposition (2.9), $t_0$ denotes the time at which the loan was initiated, and $\iota_{ik}(x^Q_{ik}) = 1 - \exp(-v_i\, x^Q_{ik})$. The choice of $\iota^Q_{ik}$ as a Poisson arrival with rate $v_i$ denotes the probability that at least one liquidator from the population $N^Q x^Q_{ik}$ detects the liquidation opportunity when detections on a protocol $i$ occur at rate $v_i$. We leave $v_i$ as a data-driven parameter to be tuned by measuring the expected time between a loan becoming undercollateralized and the corresponding request to liquidate.

**Proposition 2.9.** *Assume that a borrower on protocol $i$ takes on debt of value $d_{ik}(t_0)$ and posts collateral type $k$ of $c(t_0)$ units such that $d_{ik}(t_0)/c(t_0)p_k(t_0) < \theta_{ik}$. Assume that the liquidator always liquidates the maximum amount given by equation (4). Then for $m_{ik} = \mu_k - \frac{1}{2}\sigma_k^2 - r_i(U_i(x))$, the expected liquidation bounty paid in time $\hat{\tau}_i$ is*

$$\mathbb{E}[H(\hat{\tau}_i)|p_k(t_0)] = \frac{c(t_0)\gamma_{ik}}{d_{ik}(t_0)}\left(\frac{\theta_{ik}c(t_0)p_k(t_0)}{d_{ik}(t_0)}\right)^{\frac{m_{ik}-\sqrt{m_{ik}^2+2\sigma_k^2\omega_i}}{\sigma_k^2}}.$$

*Moreover*

$$\mathbb{E}[\hat{\tau}_i \mid p_k(t_0)] = \frac{1}{\omega_i}\left[1-\left(\frac{\theta_{ik}c(t_0)p_k(t_0)}{b(t_0)}\right)^{\frac{m_{i,k}-\sqrt{m_{i,k}^2+2\sigma_k^2\omega_i}}{\sigma_k^2}}\right].$$

Putting these components together we get that a borrower on chain $i$ with collateral type $k$ aims to minimize the expected cost

$$c_{ik}^B(a,x) = (r_i - r_f)U_i(x)\eta_{ik}^L\,\mathbb{E}[\hat{\tau}_i|p_k(t_0)] + \iota_{ik}(x_{ik}^Q)\mathbb{E}[H(\hat{\tau}_i)|p_k(t_0)]. \tag{5}$$

### 2.3.2 Lenders

The lender's aim is to minimize a cost $c_i^L(a,x)$ which depends on the expected loss due to default and the expected interest payments they receive on protocol $i$. The interest rate payments to the lender are $(r_i - \rho_i)U_i(x)\eta_i^L\,\hat{\tau}_i$. Recall from section 2.1.1 that the protocol takes a $\rho_i$ percent protocol fee of all interest rates paid. Note that a lender will be disincentivized from lending their assets on a protocol with few borrowers or a high density of lenders since the utilization rate decreases which in turn limits their interest rate payments. Additionally a lender may choose to steer clear of a protocol without many liquidators since their loans may be at risk of defaults. We model the probability of default as $\exp(-x_{ik}^Q)$, since without liquidators there would be no mechanism to recuperate a loan after it reaches the liquidation threshold. Note however that if a loan defaults then the loss is shared by all lender in the protocol. As such the expected amount lost to default on a chain $i$ is given by $\sum_{i=1}^K x_{ik}^B\eta_{ik}^B\exp(-x_{ik}^Q)$. Putting these terms together we get that a lender on protocol $i$ aims to minimize the expected cost function

$$c_i^L(a,x) = (\rho_i - r_i)U_i(x)\eta_i^L\,\mathbb{E}[\hat{\tau}_i|p_k(t_0)] + \frac{1}{N^L}\sum_{i=1}^K x_{ik}^B\eta_{ik}^B e^{-x_{ik}^Q}. \tag{6}$$

### 2.3.3 Liquidators

The liquidator's aim to minimize their cost $c_{ik}^Q(a,x)$ which depends exclusively on the expected profitability of liquidating collateral of type $k$ on protocol $i$. The liquidators profitability depends, in part, on how many many borrowers are using collateral type $k$ on protocol $i$, since the more loans are issued the more opportunities there are for liquidations to occur during the time horizon assumed in assumption (2.8). Additionally if the liquidation threshold $\theta_{ik}$ is very conservative, then liquidations occur less frequently which reduces expected liquidation profits. On the other hand, if many liquidators are monitoring the same $(i, k)$ pair, then the probability that any single liquidator successfully executes the liquidation decreases due to competition. In practice, liquidations are competitive auctions which depend on the concentration of liquidators and the liquidation bounty. In particular, higher liquidation bounties $\gamma_{ik}$ increase the reward received for liquidating bad debt, making those protocol–collateral pairs more attractive to liquidators. We model these inter-population dynamics with the cost function

$$c_{ik}^{Q}(a, x) = -N^{B} x_{ik}^{B} \frac{1}{N^{Q} x_{ik}^{Q} + 1} \mathbb{E}[H(\hat{\tau}_i) | p_k(t_0)], \tag{7}$$

where the negative cost corresponds to the payoff of the sealed bid auction described in proposition (2.6) and the expected liquidation bounty are defined in proposition (2.9).

### 2.3.4 Lower Level Game Formulation

The lower-level equilibrium can be interpreted as a generalized Nash equilibrium among three populations: borrowers, lenders, and liquidators. Each infinitesimal member of population $p \in \{B, L, Q\}$ chooses a position $s \in S_p$ so as to minimize its individual cost $c_s^p(a, x)$, while the aggregate population distribution must satisfy the simplex constraints and the shared feasibility constraints induced by the $TLV$ and utilization limits.The set of Wardrop equilibria of the population game can be characterized by the variationally inequality

$$\begin{aligned} \mathrm{SOL}(a) &= \{x \in \mathcal{X} : F(a, x)^{\top}(y - x) \geq 0 \quad \forall y \in \mathcal{X}\} \\ \mathcal{X} &= \{x : Ax = \mathbf{1}_3, \quad g(x) \leq 0\} \end{aligned} \tag{8}$$

where $g(x) = (g_{TLV}(x), g_U(x), -x)$. With $g_{TLV,i}(x) = \sum_{k=1}^{K} x_{ik}^{B} \eta_{ik}^{B} - TLV_i$ for $i = 1, \ldots, N$, $g_{U,i}(x) = N^B \sum_{k=1}^{K} \eta_{ik}^{C} \phi_{ik} x_{ik}^{B} - N^L \eta_i^L (x_i^L + \epsilon_i)$ for $i = 1, \ldots, N$, $Ax = (\sum_{s \in S_p} x_s^p)_{p \in \{B, L, Q\}}$, and

$$F(a, x) = \Big((c_s^B(a, x))_{s \in S_B}, (c_s^L(a, x))_{s \in S_L}, (c_s^Q(a, x))_{s \in S_Q}\Big).$$

## 2.4 Upper Level Game

In the upper-level game, market participants choose a set of parameters $a_i$ that minimize some cost. Based on the selected parameters of each protocol the lower-level market participants engage in a population-type game as described in problem (8). Eventually the lower-level market participants reach equilibrium distributions. They then judge their parameter choices based on two primary criteria. First, the revenue generated through fees. Typically these fees are deducted from interest payments made to lenders. Second the protocols seek to minimize the risk of failure by limiting the amount of bad debt on the protocol's balance sheet. Note that both aims are dependent on the equilibrium distribution $x$ of the lower-level game. Since if few market participants have joined a protocol then the fees they earn is small and if too few liquidators join the protocol then the risk of liquidation failure is high. We assume that each protocol has the same two aims and its its goal minimize

$$f_i(x) = \sum_{k=1}^{K} x_{ik}^B \eta_{ik}^B e^{-x_{ik}^Q} - \rho_i N^B \sum_{k=1}^{K} x_{ik}^B \eta_{ik}^B, \tag{9}$$

The first term encodes the protocol's expected loan value lost through default and the second term can be interpreted as encoding the protocol's profit. The Nash equilibrium of the upper level game is the actions $a^* = (a_1^*, \ldots, a_N^*) \in \mathcal{A}$ such that

$$a_i^* \in \operatorname*{argmin}_{a_i \in \mathcal{A}_i,\ x} f_i(x) \quad \text{s.t} \quad x \in \mathrm{SOL}(a_i, a_{-i}^*) \quad \forall i = 1, \ldots, N. \tag{10}$$

Problem (10) defines an equilibrium problem with equilibrium constraints (EPEC) (Facchinei and Pang, 2003), where the upper-level strategic interaction among protocols is coupled through the lower-level population equilibrium.

# 3 Solution Method

In practice, protocol parameter updates do not occur as a one-shot simultaneous game. Instead, protocols repeatedly adjust their parameters over time using full information about the current parameters of competing protocols. As a result, the upper-level interaction is more naturally modeled as a sequential decision process in which each protocol solves a best-response problem given the actions of others. To that end, in this section, we study problem (10) using a sequential scheme, where protocols iteratively solve their own optimization problems in response to the current state of the system. The key theoretical question is whether these dynamics converge to a reasonable solution. To answer this question, we proceed by defining the appropriate equilibrium concept for our model; variationally stationary Nash points. We then propose a domain-motivated regularization scheme and reformulate the bi-level game as regularized linkage problem as described in (Royset and Wets, 2021, Chapter 10) which allows for the application of the progressive decoupling algorithm. We conclude with a discussion regarding convergence of progressive decoupling and how a homotopy method can help recover the variationally stationary Nash point.

## 3.1 Variationally Stationary Nash Points

To define a relevant solution concept for problem (10) we first emphasize that protocols can only directly control their actions $a_i = (\lambda_{ik}, \theta_{ik}, \gamma_{ik})_{k=1}^K$. On the other hand the equilibrium of the market $x \in \mathrm{SOL}(a)$ emerges as a Wardrop equilibrium of the lower level population game which is indirectly influenced by all protocols. Therefor the economically relevant first-order condition for an arbitrary protocol $i$ is not to directly optimize over $(a_i, x)$ and seek a points $(a_i^*, x^*)$ such that $(0,0) \in \partial_{(a_i,x)} \hat{f}_i(a^*, x^*)$ and $x^* \in \mathrm{SOL}(a^*)$. Instead we propose that the the more appropriate condition is that a protocol $i$ has no first-order profitable deviation through the parameter levers it controls; i.e to find a point $a_i^*$ such that $(0, w_i) \in \partial_{(a_i,x)} \hat{f}_i(a^*, x^*)$ and $x^* \in \mathrm{SOL}(a^*)$. This distinction is important, since a protocol may prefer a different market equilibrium state without a means to achieve it. For example, it may prefer more liquidators on its platform, fewer risky borrowers, or a different allocation of lending capital but has not means of inducing such an equilibrium through small deviations in its actions. Such a equilibrium level preference is represented by a non-zero marginal term $w_i$. At the same time, the shared market equilibrium state must be stable in aggregate. Otherwise, the protocol parameters $a^*$ may be stationary only relative to a market state that is itself subject to first-order imbalance. We therefore require the shared-state sub-gradients to balance: $\sum_{i=1}^N w_i = 0$.

Economically, this means that although individual protocols may exert different marginal pressures on the lower-level equilibrium state, these pressures cancel in aggregate. For instance if the protocols collectively attempted to perturb the shared market state, there would be no first-order direction in which the equilibrium $x^*$ wishes to move. This mirrors the role of variational or normalized equilibria in games with shared constraints, where shared feasibility is represented through a common variational condition rather than through independent player-specific optimality conditions (Facchinei and Kanzow, 2007; Nabetani et al., 2011). Note that since the full EPEC is generally nonconvex, all first-order conditions are understood in the Clarke sub-differential sense.

**Definition 3.1** (Variationally Stationary Nash Point)**.** *Let*

$$\hat{f}_i(a, x) = \begin{cases} f_i(x), & a \in \mathcal{A},\ x \in \mathrm{SOL}(a), \\ +\infty, & \textit{otherwise}. \end{cases}$$

*We refer to a pair* $(a^*, x^*) \in \mathcal{A} \times X$ *as a variationally stationary Nash point with respect to* $\hat{f}_i(a, x)$ *if* $\hat{f}_i(a, x) < \infty$ *for all* $i$*, for each protocol there exists* $w_i^* \in \mathbb{R}^{2NK+N}$ *such that* $(0, w_i^*) \in \partial_{(a_i, x)} \hat{f}_i(a^*, x^*)$*, and* $\sum_{i=1}^N w_i^* = 0$.

Our solution concept is closest to the stationary Nash points introduced by Hu and Ralph (2007). In their formulation, a Nash stationary point $(a^*, x^*)$ requires each player's mathematical program with equilibrium constraints (MPEC) to be stationary with respect to both the player's own decision variable and the shared market equilibrium variable $x$. Our solution concept is therefore weaker than the Nash stationary point in the Hu-Ralph sense. Indeed a Nash stationary point is necessarily a variationally stationary Nash point.

**Proposition 3.2** (Stationary Nash implies Variationally Stationary Nash)**.** *Suppose* $(a^*, x^*) \in \mathcal{A} \times X$ *is a Nash stationary point of the EPEC in the Hu–Ralph sense. In particular, suppose that for each protocol* $i = 1, \ldots, N$*,* $(0, 0) \in \partial_{(a_i, x)} \hat{f}_i(a^*, x^*)$*. Then* $(a^*, x^*)$ *is a variationally stationary Nash point.*

## 3.2 Regularized EPEC

The full EPEC defined in equation (10) is nonlinear and generally nonconvex. Moreover the lower-level Wardrop equilibrium may be set-valued, i.e for $\mathrm{SOL}(a)$ is not necessarily unique for a fixed action profile $a$. This creates computational difficulty, since small changes in protocol parameters may cause discontinuous changes in the selected market equilibrium. To obtain a stable computational target we introduced a regularized version of the lower-level equilibrium problem (8). Specifically let $\bar{x}$ denote a reference market state, such as the current observed borrower/lender/liquidator distribution. We note that such a reference is available in a decentralized finance setting where the state of a block-chain is transparent and queriable by design. Then for some $\mu > 0$ we define $F_\mu(a, x) = F(a, x) + \mu(x - \bar{x})$. The regularized lower-level equilibrium and corresponding graph is then

$$\mathrm{SOL}_\mu(a) = \{x \in \mathcal{X} : F_\mu(a, x)^\top (y - x) \geq 0 \quad \forall y \in \mathcal{X}\}, \tag{11}$$

$$\Gamma_\mu = \{(a, x) \in \mathcal{A} \times \mathcal{X} : x \in \mathrm{SOL}_\mu(a)\} \tag{12}$$

The regularization serves the dual purpose of representing market inertia and adding curvature to the problem. Namely that borrowers,lenders, and liquidators do not instantaneously and costlessly reallocate resources across protocols in response to small small changes in liquidation parameters $(\lambda_{ik}, \theta_{ik}, \gamma_{ik})$, since moving from one protocol-collateral pair $(i, k)$ may involve gas costs, smart contract approval, search costs, and other market frictions. We observe that the lower level problem (11) admits a solution, although we note that the solution is not necessarily unique.

**Proposition 3.3** (Existence and compactness of the regularized lower-level problem)**.** *Assume $\bar{x} \in \mathcal{X}$. Then for every $\mu \geq 0$ and every $a \in \mathcal{A}$, the regularized lower-level equilibrium set $\mathrm{SOL}_\mu(a)$ is nonempty. Moreover, the graph $\Gamma_\mu$ is compact.*

The same principle is applied at the protocol level. Specifically we let $\bar{a}$ denote a the current or baseline parameter vector and augment the protocol cost functions as

$$\hat{f}_i^{\mu,\beta}(a, x) = \begin{cases} f_i(x) + \frac{\beta_i}{2}\|a_i - \bar{a}_i\|^2 & (a, x) \in \Gamma_\mu \\ \infty & \text{otherwise,} \end{cases} \tag{13}$$

with $\beta \in (0, \infty)^N$. In this case the regularization term encodes the fact that large and sudden changes in protocol parameters may induce governance costs or user confusion. For example consider the setting where a user currently has a debt to collateral ratio well bellow the threshold $\theta_{ik}$. A large decrease in $\theta_{ik}$ may instantaneously render many loans liquidatable, which may be unattractive to borrowers and pose a “flight risk” of capital. The regularization terms encode economically meaningful frictions, while also producing the regularized problem of finding a variationally stationary Nash point with respect to $\hat{f}_i^{\mu,\beta}(a, x)$.

### 3.3 Regularized EPEC as a Linkage Problem

The regularized EPEC remains difficult to solve directly because each protocol’s stationary conditions are coupled through the shared lower-level market equilibrium. Linkage problems as described by Royset and Wets (2021) provide a useful framework for representing this type of coupled equilibrium system. The main idea behind formulating the EPEC as a linkage problem is that each protocol stores its own local copy of the full action profile and the lower level market state, then a linkage constraint requiring all copies to agree is imposed. In our setting, protocol $i$’s copy is denoted as $z_i = (a^i, x^i)$, where $a^i = (a_1^i, \ldots, a_N^i)$ is protocol $i$’s copy of the full protocol parameter profile and $x^i$ is its copy of the lower-level population equilibrium. Agreement among the protocols’ copies is imposed by enforcing that $z = (z_1, \ldots, z_N) \in \mathcal{L}$, where

$$\mathcal{L} = \{(z_1, \ldots, z_N) : z_1 = \cdots = z_N\}. \tag{14}$$

The task of finding a variationally stationary Nash point of the regularized problem can then be formulated as

$$z \in \mathcal{L}, \quad y \in \mathcal{L}^{\perp}, \quad y \in S^{\mu,\beta}(z), \tag{15}$$

where the orthogonal compliment of $\mathcal{L}$ is defined as $\mathcal{L}^{\perp} = \{(y_1, \ldots, y_N) : y_1 + \cdots + y_N = 0\}$ by lemma A.1 and $S^{\mu,\beta}(z) = S_1^{\mu,\beta}(z_1) \times \cdots \times S_N^{\mu,\beta}(z_N)$ with

$$S_i^{\mu,\beta}(z_i) = \Big\{(0, \ldots, v_i, \ldots, 0, w_i) : (v_i, w_i) \in \partial_{(a_i,w_i)} \hat{f}_i^{\mu,\beta}(a^i, x^i)\Big\}.$$

**Proposition 3.4** (Linkage solutions and variationally stationary Nash points)**.** *A pair* $(z^*, y^*)$ *solves the linkage problem* (15) *if and only if the common pair* $(a^*, x^*)$ *induced by* $z^* \in \mathcal{L}$ *is a variationally stationary Nash point with respect to* $\hat{f}_i^{\mu,\beta}(a, x)$*.*

Proposition 3.4 shows that solving the linkage problem is equivalent to identifying a variationally stationary Nash point of the regularized problem. It motivates the use of the progressive decoupling algorithm.

## 3.4 Graphical Approximation and Progressive Decoupling

We now describe the numerical scheme used to approximate a variationally stationary Nash point of EPEC. The method has two layers. The outer layer is a graphical approximation algorithm wherein the regularized linkage problem is solved iteratively for a sequence of $\{(\mu^\nu, \beta^\nu)\}$ that converges to $(0, 0)$. We view this sequence as defining a homotopy from a more regularized problem to the original EPEC. Indeed this approach mirrors classical homotopy methods, where one solves an easier problem first and then gradually deforms it into the target problem while warm-starting each new solve from the previous solution. The same principle is used here. For larger values of $\mu$ and $\beta$, the regularized linkage problem which uses $S^{\mu,\beta}$ is expected to have better local stability and better conditioning.

However, to apply the homotopy method, we must have a means of recovering solutions to the linkage problem for any $(\mu^\nu, \beta^\nu)$. To do this we apply a progressive decoupling algorithm. Progressive decoupling is an algorithm first proposed by Rockafellar (2018) which operates by temporarily relaxing the linkage constraint encoded by $\mathcal{L}$, and solving the decoupled local problems, and then projecting the resulting local copies back onto the agreement subspace $\mathcal{L}$. The steps of the process can be derived for parameters $\theta > 0$ and $\kappa > \theta$ and has three steps. We let $\nu$ be the iteration counter of the homotopy loop and $r$ be the iteration counter of the progressive decoupling iterations.

In Step 1 we solve for an intermediate point $\hat{z}^r$ satisfying the condition $y^r \in S^{\mu,\beta}(\hat{z}^r) + \kappa(\hat{z}^r - z^r)$. In other words we temporarily ignore the linkage constraint $z^r \in \mathcal{L}$ so that each protocol can compute its own action and optimal lower level equilibrium point. Recall that $S^{\mu,\beta}(z) = S_1^{\mu,\beta}(z_1) \times \cdots \times S_N^{\mu,\beta}(z_N)$, so the earlier Step 1 condition is equivalent to solving $N$ independant generalized equations of the form

$$y_i^r \in S_i^{\mu,\beta}(\hat{z}_i^r) + \kappa(\hat{z}_i^r - z_i^r), \qquad i = 1, \ldots, N. \tag{16}$$

From a computational point of view its useful to observe that the subproblems described in equation (16) are the first order optimality conditions of the problems

$$\hat{z}_i^r \in \operatorname*{argmin}_{z_i} \left\{ \psi_i^{\mu,\beta}(z_i) - \langle y_i^r, z_i \rangle + \frac{\kappa}{2}\|z_i - z_i^r\|^2 \right\}. \tag{17}$$

When the local problem is convex, (17) is a genuine minimization problem. When the local problem is nonconvex, the same formula is interpreted as a proximal stationarity subproblem, and we compute a stationary point satisfying (16). The two extra terms in (17) serve important roles. Specifically the affine term $-\langle y_i^r, z_i \rangle$ encodes the current dual pressure associated with disagreement among the local copies. Whereas the quadratic term $\frac{\kappa}{2}\|z_i - z_i^r\|^2$ keeps the new local proposal close to the previous consensus iterate. In this way, Step 1 can be interpreted as asking each protocol to compute a regularized local best response to the current dual prices. After Step 1, the intermediate vector $\hat{z}^r = (\hat{z}_1^r, \dots, \hat{z}_N^r)$ will generally not satisfy the agreement constraint (i.e $\hat{z}^r$ is not necessarily a member of $\mathcal{L}$). So naturally step 2 projects $\hat{z}^r$ onto the linkage subspace by computing $z^r = \operatorname{proj}_{\mathcal{L}}(\hat{z}^r)$. Since $\mathcal{L} = \{(z_1, \dots, z_N) : z_1 = \cdots = z_N\}$ the projection is the average of the local copies,

$$z_i^{r+1} = \frac{1}{N}\sum_{j=1}^{N} \hat{z}_j^r, \qquad i = 1, \dots, N. \tag{18}$$

Finally, Step 3 updates the dual variable as $y^{r+1} = y^r - (\kappa - \theta)\operatorname{proj}_{\mathcal{L}^\perp}(\hat{z}^r)$. Using the identity that $\hat{z}^r = \operatorname{proj}_{\mathcal{L}}(\hat{z}^r) + \operatorname{proj}_{\mathcal{L}^\perp}(\hat{z}^r)$ and that fact that $z^{r+1} = \operatorname{proj}_{\mathcal{L}}(\hat{z}^r)$ we can rewrite the dual update without a projection operation as

$$y_i^{r+1} = y_i^r - (\kappa - \theta)(\hat{z}_i^r - z_i^{r+1}), \qquad i = 1, \dots, N. \tag{19}$$

The dual update can be interpreted as measuring each protocol's deviation from the consensus copy. We note that $y_i^{r+1}$ is indeed a member of $\mathcal{L}^\perp$ since

$$\sum_{i=1}^{N} y_i^{r+1} = \sum_{i=1}^{N} y_i^r - (\kappa - \theta)\sum_{i=1}^{N}(\hat{z}_i^r - z_i^{r+1}) = \sum_{i=1}^{N} y_i^r - (\kappa - \theta)\left(\sum_{i=1}^{N}\hat{z}_i^r - N\left(\frac{1}{N}\sum_{j=1}^{N}\hat{z}_j^r\right)\right) = 0.$$

The full algorithm with the homotopy outer loop and progressive decoupling inner loop can be writen compactly as follows.

**Algorithm 1** Homotopy Progressive Decoupling

```
Choose sequences $\mu^\nu \downarrow 0$, $\beta^\nu \downarrow 0$, and $\varepsilon^\nu \downarrow 0$.
Initialize with an initial pair $z^0 \in \mathcal{L}$, $y^0 \in \mathcal{L}^\perp$.
for $\nu = 1, 2, \dots$ do
    Set $S^\nu = S^{\mu^\nu, \beta^\nu}$ and $f_i^\nu = \hat{f}_i^{\mu^\nu, \beta^\nu}$.
    Choose $\theta^\nu > 0$ and $\kappa^\nu > \theta^\nu$.
    Initialize the inner loop with $z^{\nu,0} = z^{\nu-1}$, $y^{\nu,0} = y^{\nu-1}$.
    while dist $\left(y^{\nu,r+1}, S^\nu(z^{\nu,r+1})\right) \le \varepsilon^\nu$ do
        Compute $\hat{z}_i^{\nu,r} \in \operatorname{argmin}_{z_i}\{f_i^\nu(z_i) - \langle y_i^{\nu,r}, z_i\rangle + \frac{\kappa^\nu}{2}\|z_i - z_i^{\nu,r}\|^2\}, \quad i = 1, \dots, N$.
        Set $z_i^{\nu,r+1} = \frac{1}{N}\sum_{j=1}^N \hat{z}_j^{\nu,r}, \quad i = 1, \dots, N$.
        Set $y_i^{\nu,r+1} = y_i^{\nu,r} - (\kappa^\nu - \theta^\nu)\left(\hat{z}_i^{\nu,r} - z_i^{\nu,r+1}\right), \quad i = 1, \dots, N$.
    end while
    Set $z^\nu = z^{\nu,r+1}, \quad y^\nu = y^{\nu,r+1}$.
end for

return a cluster point of $\{(z^\nu, y^\nu)\}$.
```

Indeed the cluster points of the sequence generated by algorithm (1) will be variationally stationary Nash points of the original EPEC. The practical interpretation of Algorithm (1) is that it gives protocols a means of updating their parameters in response to the current market state and the actions of competing protocols. In this way, the algorithm provides a decentralized adjustment rule by which protocols can update liquidation thresholds, collateral factors, and liquidation bounties until no protocol has a first-order profitable parameter deviation, while the induced lower-level market equilibrium remains stationary in aggregate.

**Theorem 3.5** (Convergence)**.** *Let* $\{(z^\nu, y^\nu)\}$ *be generated by algorithm* (1) *with* $(\mu^\nu, \beta^\nu) \to (0, 0)$*. All cluster points of* $\{(z^\nu, y^\nu)\}$ *will be the variationally stationary Nash points with respect to* $\hat{f}_i$ *if* $\operatorname{LimOut} \operatorname{gph} S^{\mu^\nu, \beta^\nu} \subset \operatorname{gph} S^{0,0}$ *and for all* $\nu$ *there exists a* $\theta > 0$ *such that* $S^{\mu^\nu, \beta^\nu} + \theta \operatorname{proj}_{\mathcal{L}^\perp}$ *is maximally monotone and the regularized linkage problem* (15) *has a solution for all* $\mu, \beta > 0$.

## 4 Numerical Experiments

We evaluate the proposed update rule on a synthetic lending market with $N = 100$ protocols and $K = 50$ collateral assets. This produces $NK = 5000$ protocol-collateral pairs. The purpose of our experiment is to answer the question; if protocols use the proposed algorithm can they reduce liquidation bounties below the commonly used $5\%$ benchmark while preserving a financially safe lower level market equilibrium with few defaults and stable revenue through the taxation of liquidation events.

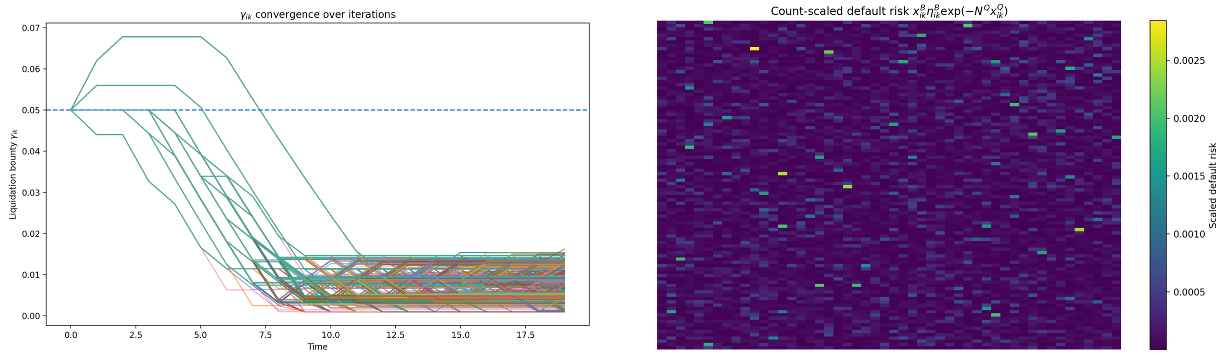
We initialize the collateral factors around at $0.75$, liquidation thresholds at $0.85$, and set all initial liquidation bounties equal to the static benchmark value $0.05$. Small standard normal perturbations are added to the initial collateral factors and liquidation thresholds to avoid a completely symmetric market. We choose $\underline{\lambda} = 0.5$ and $\underline{\theta} = 0.70$ which characterizes the action space described in section. The collateral assets follow the geometric Brownian motion model described in section 2.1. We normalize all initial collateral prices to $p_k(t_0) = 1$ and set the drifts $\mu_k$ are linearly spaced between $-1$ and $1$. Similarly asset volatilities are linearly spaced between $0.35$ and $0.95$, so that the synthetic market contains collateral assets ranging from relatively low-risk to high-risk.

The market contains $N^B = 80{,}000$ borrowers, $N^L = 2{,}000$ lenders, and $N^Q = 300$ liquidators. The average lender deposit is set to $\eta_i^L = 100$, while the average posted collateral is normalized to $\eta_{ik}^C = 1$. The initial borrower leverage satisfies $\phi_{ik} = 0.90\lambda_{ik}$, so that borrower leverage remains feasible as collateral factors are updated. Each protocol has a total-loan-value cap of $16,000$ and seed liquidity $\epsilon_i = 0.025$, which prevents the utilization rate from becoming discontinuous when lender mass is small. Borrow interest rates are computed using the smoothed Aave-style kinked rate function with base rate $r_i^0 = 0.01$, low-utilization slope $r_i^1 = 0.0354$, high-utilization slope $r_i^2 = 0.45$, target utilization $U_i^{\text{opt}} = 0.80$, and smoothing parameter $\delta = 10^{-3}$. The risk-free rate is fixed at the standard $r_f = 0.02$, and the protocol fee is set to $\rho_i = 0.005$. Again we introduce mild protocol-level heterogeneity by perturbing these quantities using small standard normal random multiplicative factors for each protocol.

We use algorithm 1 where the progressive decoupling step and graphical approximation outer loop is run for at most 1000 iterations. We run the algorithm for 20 iterations which represents each protocol making 20 action updates. We evaluate the resulting equilibrium using three main classes of metrics. First, we track the liquidation bounty distribution over time. Second, we evaluate financial safety. Specifically we report the count-scaled default exposure

$$D_{\text{scaled}} = \sum_{i=1}^{N}\sum_{k=1}^{K} x_{ik}^B \eta_{ik}^B \exp(-N^Q x_{ik}^Q), \tag{20}$$

which penalizes protocol-asset pairs with borrower exposure but low liquidator coverage. We report the results in figure 1.



**(a)** Liquidation bounty over the 20-move simulation. Each curve tracks the liquidation bounty $\gamma_{ik}$ for each protocol-collateral pair.

**(b)** Default risk across protocol-collateral pairs at the $20^{th}$ action. Each cell represent a protocol-collateral pair, with the Y axis as the protocol and X axis as the collateral

**Figure 1:** Evolution of liquidation incentives and default risk under the proposed EPEC-based controller.

The results show that the protocols are able to gradually reduce liquidation bounties over the 20-move simulation. This indicates that the initial bounty levels are not necessarily cost-efficient. Instead by accounting for the equilibrium response of borrowers, lenders, and liquidators, the protocols can lower the direct incentive (liquidation bounty $\gamma$) paid to liquidators without creating unsafe market conditions with a high volume of defaults. Indeed, the default-risk heatmap remains low across protocol-collateral pairs. This indicates that the decrease in liquidation bounties is not achieved by shifting risk onto the system. Instead the equilibrium-based updates appear to identify parameter regimes in which liquidator participation remains sufficient despite lowered liquidation bounties. This answers the primary research question in the affirmative. Namely that protocols can use algorithmic controllers to dynamically adjust liquidation bounties in order to reduce effective borrower fees while still retaining sufficient market participation of liquidators.

## 5 Discussion & Limitations

This paper proposes a game-theoretic framework for modeling competition among over-collateralized lending protocols and for dynamically updating liquidation parameters using a homotopy progressive decoupling algorithm. The main modeling contribution is to treat liquidation as an endogenous market interaction among borrowers, lenders, and liquidators, rather than as a fixed exogenous mechanism. In this framework, protocols choose liquidation thresholds, collateral factors, and liquidation bounties, while market participants respond through a lower-level Wardrop equilibrium. The resulting equilibrium problem with equilibrium constraints captures the feedback loop between protocol design, market participation, and liquidations. The second contribution is the solution concept of variationally stationary Nash points, which is intentionally weaker than a classical Nash stationary point. The new solution concept reflect that fact that each protocol must have no first-order profitable deviation in its own controlled parameters, while the sub-gradients on the shared market state must only balance on aggregate. This gives a tractable first-order equilibrium concept that is aligned with the reformulation of the EPEC to a linkage problem. There are several important limitations. First the full EPEC is non-convex, which means that the regularized linkage problem is not guaranteed to be globally elicitable maximal monotone, although the regularization is incorporated to add curvature to the problem. Second we do not prove existence of a variationally stationary Nash point of the full EPEC, and instead only guarantee that lower level equilibirum exists. Third the algorithm converges to variationally stationary Nash points which is not necessarily a globally optimal action profile. Despite these limitations the framework provides a useful testbed for studying algorithmic control of liquidation-focused protocol parameters.

# References

Aave (2020). Aave protocol whitepaper.

Bastankhah, M., Nadkarni, V., Wang, X., Jin, C., Kulkarni, S., and Viswanath, P. (2024a). Thinking fast and slow: Data-driven adaptive defi borrow-lending protocol.

Bastankhah, M., Nadkarni, V., Wang, X., and Viswanath, P. (2024b). Agilerate: Bringing adaptivity and robustness to defi lending markets.

Chiu, J. and Danisman, F. (2026). DeFi lending: Returns, leverage, and liquidation risk. Staff Analytical Paper 2026-13, Bank of Canada.

ethereum.org (2026). Blocks. Accessed: 2026-05-11.

Facchinei, F. and Kanzow, C. (2007). Generalized nash equilibrium problems. *4OR: A Quarterly Journal of Operations Research*, 5(3):173–210.

Facchinei, F. and Pang, J.-S. (2003). *Finite-Dimensional Variational Inequalities and Complementarity Problems*. Springer, New York.

Hu, X. and Ralph, D. (2007). Using epecs to model bilevel games in restructured electricity markets with locational prices. *Operations Research*, 55(5):809–827.

Krishna, V. (2010). Chapter one - introduction. In Krishna, V., editor, *Auction Theory (Second Edition)*, pages 1–8. Academic Press, San Diego, second edition edition.

Li, Z., Li, J., He, Z., Luo, X., Wang, T., Ni, X., Yang, W., Chen, X., and Chen, T. (2023). Demystifying defi mev activities in flashbots bundle. In *Proceedings of the 2023 ACM SIGSAC Conference on Computer and Communications Security*, CCS '23, page 165–179, New York, NY, USA. Association for Computing Machinery.

Nabetani, K., Tseng, P., and Fukushima, M. (2011). Parametrized variational inequality approaches to generalized nash equilibrium problems with shared constraints. *Computational Optimization and Applications*, 48(3):423–452.

Perez, D., Werner, S. M., Xu, J., and Livshits, B. (2021). *Liquidations: DeFi on a Knife-Edge*, page 457–476. Springer Berlin Heidelberg.

Qin, K., Zhou, L., Gamito, P., Jovanovic, P., and Gervais, A. (2021). An empirical study of defi liquidations: Incentives, risks, and instabilities.

Rockafellar, R. T. (2018). Progressive decoupling of linkages in optimization and variational inequalities with elicitable convexity or monotonicity. *Set-Valued and Variational Analysis*, 27(4):863–893.

Royset, J. O. and Wets, R. J.-B. (2021). *An Optimization Primer*. Springer.

# A Proofs

## A.1 Proof of Lemma 2.3

Suppose that the borrower has debt value $d_{ik}(t)$ and posts $c(t)$ units of collateral with price $p_k(t)$. Let $\ell$ denote the value of debt repaid by the liquidator. Since the liquidator receives collateral of value $(1+\gamma_{ik})\ell$, the number of collateral units removed is $(1+\gamma_{ik})\ell/p_k(t)$. Hence

$$d'_{ik}(t) = d_{ik}(t) - \ell, \qquad c'(t) = c(t) - \frac{(1+\gamma_{ik})\ell}{p_k(t)}.$$

The post-liquidation debt-to-collateral ratio is therefore

$$\frac{d'_{ik}(t)}{c'(t)p_k(t)} = \frac{d_{ik}(t) - \ell}{c(t)p_k(t) - (1+\gamma_{ik})\ell}.$$

To restore the position to the liquidation threshold, we require

$$\frac{d_{ik}(t) - \ell}{c(t)p_k(t) - (1+\gamma_{ik})\ell} \leq \theta_{ik}.$$

Assuming $c(t)p_k(t) - (1+\gamma_{ik})\ell > 0$, this is equivalent to

$$d_{ik}(t) - \ell \leq \theta_{ik} c(t)p_k(t) - \theta_{ik}(1+\gamma_{ik})\ell,$$

or

$$d_{ik}(t) - \theta_{ik} c(t)p_k(t) \leq \ell\big(1 - \theta_{ik}(1+\gamma_{ik})\big).$$

Thus, if $1 - \theta_{ik}(1+\gamma_{ik}) > 0$, the minimum liquidation value is

$$\ell_{\min}(t) = \max\left\{0,\ \frac{d_{ik}(t) - \theta_{ik} c(t)p_k(t)}{1 - \theta_{ik}(1+\gamma_{ik})}\right\}.$$

## A.2 Proof of Proposition 2.6

Let $N := N^Q_{ik}$. In a first-price auction with $N$ risk-neutral bidders whose private values are i.i.d. uniformly distributed on $[0, W_{ik}]$, the unique symmetric equilibrium bid function is

$$\beta(v) = \frac{N-1}{N} v$$

as given in example 2.1 by Krishna (2010). Let $V_{(N)} = \max\{V_1, \ldots, V_N\}$ denote the highest valuation. Since the bidder with the highest valuation wins, the winner's realized net reward is

$$V_{(N)} - \beta(V_{(N)}) = V_{(N)} - \frac{N-1}{N} V_{(N)} = \frac{1}{N} V_{(N)}.$$

Since $V_i \sim \text{Uniform}([0, W_{ik}])$, the expected maximum order statistic is $\mathbb{E}[V_{(N)}] = NW_{ik}/(N+1)$. Therefore,

$$\mathbb{E}\big[V_{(N)} - \beta(V_{(N)})\big] = \frac{1}{N}\mathbb{E}[V_{(N)}] = \frac{1}{N} \cdot \frac{N}{N+1}W_{ik} = \frac{1}{N+1}W_{ik}.$$

Hence the expected net reward of the winning liquidator is $R_{ik}(N_{ik}^L) = W_{ik}/(N_{ik}^L + 1)$.

## A.3 Proof of Proposition 2.9

Our aim is to find

$$\begin{aligned}
&\mathbb{E}[H(\tau \wedge \tau_\theta)|p_k(t_0)] = \\
&\mathbb{E}[H(\tau)|\tau_\theta \le \tau, p_k(t_0)]\mathbb{P}(\tau_\theta \le \tau|p_k(t_0)) \\
&\text{where,} \quad \tau_\theta = \inf_t \Big\{t \ge t_0 : \frac{b(t)}{c(t_0)p_k(t)} \ge \theta\Big\}
\end{aligned}$$

To do this, we first find $\mathbb{P}(\tau_\theta \le \tau|p_k(t_0))$ then find the expectation. To find the probability, consider the function $X(t) = \ln(c(t_0)p_k(t)/b(t))$.Using the fact that $b(t) = b(t_0)e^{r_i(U_i)(t-t_0)}$ and the GBM $p_k(t) = p(t_0)\exp\{(\mu_k - (1/2)\sigma_k^2)(t-t_0) - \sigma_k(W_t^k - W_{t_0}^k)\}$ we apply Itô's formula to get

$$dX(t) = m_{i,k}dt + \sigma_k dW_t^k, \quad m_{i,k} = \mu_k - \frac{1}{2}\sigma_k^2 - r_i(U_i).$$

The problem of finding $\mathbb{P}(\tau_\theta \le \tau)$ then simplifies to find a function $f(x)$ defined as

$$\mathbb{P}(\tau_\alpha \le \tau|X(t_0) = x) = \mathbb{E}[\mathbb{P}(\tau_\alpha \le \tau|\tau_\alpha)|X(t_0) = x] = \mathbb{E}[e^{-\omega_i(\tau_\alpha - t_0)}|X(t_0) = x] := f(x)$$

where $x = \ln(c(t_0)p_k(t_0)/b(t_0))$, $\tau_\alpha = \inf_t\{t \ge t_0 : X(t) \le \alpha := \ln(1/\theta)\}$, and $X(t)$ satisfies the SDE $dX(t) = m_{i,k}dt + \sigma_k dW_t^k$. Under these conditions $f(x)$ satisfies the Feynman-Kac formula

$$\frac{1}{2}\sigma_k^2 f'' + m_{i,k}f' - \omega_i f = 0, \quad \text{s.t} \quad f(a) = 1, \lim_{x\to\infty} f(x) = 0.$$

Trying the ansatz of $f(x) = \exp(-\kappa(x - \alpha))$ yields

$$\mathbb{P}(\tau_\alpha \le \tau|X(t_0) = x) = f(x) = \exp\left\{\frac{m_{i,k} - \sqrt{m_{i,k}^2 + 2\sigma_k^2\omega_i}}{\sigma_k^2}(x - \alpha)\right\},$$

which satisfies the boundary conditions. Whats left is to find is $\mathbb{E}[H(\tau_\alpha)|X(t_0) = x, \tau_\alpha \le \tau]$. Note that $\ell_{\min}(\tau_\alpha) = 0$ since at $\tau_\alpha$, $X(\tau_\alpha) = \ln(c(\tau_\alpha)p_k(\tau_\alpha)/b(\tau_\alpha)) = \alpha = \ln(1/\theta)$ which implies that $d_{i,k}(t) = \theta_{i,k}c(t)p_k(t)$. Therefore $H(\tau_\alpha)$ is deterministically given as $c(t_0)\chi(h(\tau_\alpha))(1+\gamma) = c(t_0)\chi(1)(1+\gamma) =$

$(c(t_0)/2)(1+\gamma)$. So we conclude that

$$\begin{aligned}\mathbb{E}[H(\tau\wedge\tau_\alpha)|X(t_0)=x] &= \frac{c(t_0)(1+\gamma)}{2}\exp\left\{\frac{m_{i,k}-\sqrt{m_{i,k}^2+2\sigma_k^2\omega_i}}{\sigma_k^2}(x-\alpha)\right\}\\ &= \frac{c(t_0)(1+\gamma)}{b(t_0)}\left(\frac{\theta_{ik}c(t_0)p_k(t_0)}{b(t_0)}\right)^{\frac{m_{i,k}-\sqrt{m_{i,k}^2+2\sigma_k^2\omega_i}}{\sigma_k^2}}\end{aligned}$$

Moreover, the same calculation gives a closed-form expression for the expected effective time horizon $\hat{\tau}_i := \tau\wedge\tau_\alpha$. Indeed, conditioning on $\tau_\alpha$, and using the independence of $\tau\sim\text{Exp}(\omega_i)$, we have

$$\begin{aligned}\mathbb{E}[\tau\wedge\tau_\alpha \mid X(t_0)=x] &= \mathbb{E}\left[\mathbb{E}[\tau\wedge\tau_\alpha \mid \tau_\alpha, X(t_0)=x] \mid X(t_0)=x\right]\\ &= \mathbb{E}\left[\int_{t_0}^{\tau_\alpha}\mathbb{P}(\tau>s \mid \tau\geq t_0)\,ds \mid X(t_0)=x\right].\end{aligned}$$

Since $\tau-t_0\sim\text{Exp}(\omega_i)$, we have $\mathbb{P}(\tau>s\mid\tau\geq t_0)=e^{-\omega_i(s-t_0)}$. Therefore,

$$\begin{aligned}\mathbb{E}[\tau\wedge\tau_\alpha \mid X(t_0)=x] &= \mathbb{E}\left[\int_{t_0}^{\tau_\alpha}e^{-\omega_i(s-t_0)}\,ds \mid X(t_0)=x\right]\\ &= \mathbb{E}\left[\frac{1-e^{-\omega_i(\tau_\alpha-t_0)}}{\omega_i} \mid X(t_0)=x\right]\\ &= \frac{1-\mathbb{E}[e^{-\omega_i(\tau_\alpha-t_0)} \mid X(t_0)=x]}{\omega_i}.\end{aligned}$$

Using the expression derived above, $\mathbb{E}[e^{-\omega_i(\tau_\alpha-t_0)} \mid X(t_0)=x]=\exp\left\{\frac{m_{i,k}-\sqrt{m_{i,k}^2+2\sigma_k^2\omega_i}}{\sigma_k^2}(x-\alpha)\right\}$, we obtain

$$\mathbb{E}[\hat{\tau}_i \mid p_k(t_0)] = \frac{1}{\omega_i}\left[1-\left(\frac{\theta_{ik}c(t_0)p_k(t_0)}{b(t_0)}\right)^{\frac{m_{i,k}-\sqrt{m_{i,k}^2+2\sigma_k^2\omega_i}}{\sigma_k^2}}\right].$$

### A.4 Proop of Proposition 3.2

Fix a protocol $i$. Given the rival protocols' actions $a_{-i}^*$, protocol $i$'s MPEC is

$$\underset{a_i,x}{\text{minimize}}\quad f_i(x)\quad \text{s.t}\quad a_i\in A_i,\quad x\in\text{SOL}(a_i,a_{-i}^*).$$

Equivalently, using the extended-value representation, define

$$\hat{f}_i(a_i,a_{-i}^*,x)=\begin{cases} f_i(x) & a_i\in A_i,\ x\in\text{SOL}(a_i,a_{-i}^*),\\ +\infty & \text{otherwise.}\end{cases}$$

Then protocol $i$'s MPEC is equivalent to the unconstrained extended-value problem $\text{minimize}_{a_i,x}\ \hat{f}_i(a_i, a^*_{-i}, x)$. By assumption, $(a^*_i, x^*)$ is a local minimizer of this extended-value problem. The Clarke-Fermat necessary condition for local minimality gives $(0, 0) \in \partial_{(a_i,x)}\hat{f}_i(a^*_i, a^*_{-i}, x^*)$. Because $(a^*_i, x^*)$ is feasible for protocol $i$'s MPEC, we also have $x^* \in \mathrm{SOL}(a^*_i, a^*_{-i}) = \mathrm{SOL}(a^*)$. Since this holds for every protocol $i$, the common lower-level equilibrium condition $x^* \in \mathrm{SOL}(a^*)$ is satisfied. Now define $w^*_i := 0$ for all $i = 1, \dots, N$. Then, for every protocol $i$, $(0, w^*_i) = (0, 0) \in \partial^C_{(a_i,x)}\hat{f}_i(a^*, x^*)$. Moreover,

$$\sum_{i=1}^{N} w^*_i = \sum_{i=1}^{N} 0 = 0.$$

Thus $x^* \in \mathrm{SOL}(a^*)$, each protocol satisfies. Therefore $(a^*, x^*)$ is a variationally stationary Nash point.

## A.5 Proof of Proposition 3.3

By construction, $\mathcal{A} = \prod_{i=1}^{N} A_i$ where $A_i = ([\underline{\lambda}, 1] \times [\underline{\theta}, 1] \times [0, 1])^K$ so $\mathcal{A}$ is compact. By assumption, $X \neq \emptyset$. Moreover, $\mathcal{X}$ is convex and closed because it is defined by affine equality and inequality constraints. The simplex constraints imply that every component of $x^B, x^L$, and $x^Q$ lies in $[0, 1]$, so $\mathcal{X}$ is bounded. So $\mathcal{X}$ is nonempty, compact, and convex. We next note that $F_\mu$ is continuous on $\mathcal{A} \times \mathcal{X}$. Indeed, the utilization rate is continuous because $x^L_i + \epsilon_i \geq \epsilon_i > 0$, the interest-rate function is smoothed, the lower bound $\underline{\theta} > 0$ keeps the expected liquidation expression finite, and the regularization term $\mu(x - \bar{x})$ is continuous.

Fix $a \in \mathcal{A}$. Since $X$ is nonempty, compact, and convex, and $x \mapsto F_\mu(a, x)$ is continuous, the finite-dimensional variationally inequality

$$F_\mu(a, x)^\top (y - x) \geq 0 \quad \forall y \in \mathcal{X}$$

admits a solution by (Facchinei and Pang, 2003, Corollary 2.2.5), so $\mathrm{SOL}_\mu(a) \neq \emptyset$. It remains to prove that $\Gamma_\mu$ is compact. Since $\Gamma_\mu \subseteq \mathcal{A} \times \mathcal{X}$, and $\mathcal{A} \times X$ is compact, it suffices to show that $\Gamma_\mu$ is closed. To that end let

$$(a^n, x^n) \in \Gamma_\mu, \qquad (a^n, x^n) \to (a, x).$$

For every $y \in \mathcal{X}$, $F_\mu(a^n, x^n)^\top (y - x^n) \geq 0$. Passing to the limit using continuity of $F_\mu$ gives

$$F_\mu(a, x)^\top (y - x) \geq 0 \qquad \forall y \in \mathcal{X}.$$

Therefor $x \in \mathrm{SOL}_\mu(a)$, and as a result $(a, x) \in \Gamma_\mu$. Hence $\Gamma_\mu$ is closed. Since it is a closed subset of the compact set $\mathcal{A} \times \mathcal{X}$, $\Gamma_\mu$ is compact.

## A.6 Proof of Lemma A.1

**Lemma A.1** (Orthogonal complement of the agreement subspace)**.** *Let* $\mathcal{L} = \{(z_1, \dots, z_N) : z_1 = \cdots = z_N\}$ *be the agreement subspace. Then* $\mathcal{L}^\perp = \left\{(y_1, \dots, y_N) : \sum_{i=1}^{N} y_i = 0\right\}$.

*Proof.* First, suppose $y = (y_1, \dots, y_N)$ satisfies $\sum_{i=1}^{N} y_i = 0$. For any $z \in \mathcal{L}$, there exists a common vector

$\bar{z}$ such that $z = (\bar{z}, \dots, \bar{z})$. Therefore,

$$\langle y, z \rangle = \sum_{i=1}^{N} \langle y_i, \bar{z} \rangle = \left\langle \sum_{i=1}^{N} y_i, \bar{z} \right\rangle = 0.$$

Therefor $y \in \mathcal{L}^{\perp}$. For the other direction, suppose $y \in \mathcal{L}^{\perp}$. Then for every $\bar{z}$ we have that $(\bar{z}, \dots, \bar{z}) \in \mathcal{L}$. Therefor

$$0 = \langle y, (\bar{z}, \dots, \bar{z}) \rangle = \sum_{i=1}^{N} \langle y_i, \bar{z} \rangle = \left\langle \sum_{i=1}^{N} y_i, \bar{z} \right\rangle.$$

Since this holds for every $\bar{z}$, it follows that $\sum_{i=1}^{N} y_i = 0$. Therefore, $\mathcal{L}^{\perp} = \left\{ (y_1, \dots, y_N) : \sum_{i=1}^{N} y_i = 0 \right\}$.
□

### A.7 Proof of Proposition 3.4

Suppose first that $(z^*, y^*)$ solves the regularized linkage problem (15). Since $z^* \in \mathcal{L}$, there exists a common pair $(a^*, x^*)$ such that $z_i^* = (a^*, x^*)$ for all $i = 1, \dots, N$. Since $y^* \in S^{\mu,\beta}(z^*)$, for each protocol $i$, there exists

$$(v_i^*, w_i^*) \in \partial_{(a_i,x)} \hat{f}_i^{\mu,\beta}(a^*, x^*)$$

such that $y_i^* = (0, \dots, v_i^*, \dots, 0, w_i^*)$. Moreover, since $y^* \in \mathcal{L}^{\perp}$, we have that $\sum_{i=1}^{N} y_i^* = 0$. Looking at the $a_j$-block of this equality, only $y_j^*$ contributes a possibly nonzero action component, therefor $v_j^* = 0$ for all $i = 1, \dots, N$. Looking at the shared $x$-block gives $\sum_{i=1}^{N} w_i^* = 0$. Finally, because

$$(0, w_i^*) \in \partial_{(a_i,x)} \hat{f}_i^{\mu,\beta}(a^*, x^*)$$

for each $i$ (i.e the clark sub-differential is non-empty), the point $(a^*, x^*)$ must lie in the domain of $\hat{f}_i^{\mu,\beta}$. Therefore $(a^*, x^*) \in \Gamma_\mu$, which in turn implies that

$$x^* \in \mathrm{SOL}_\mu(a^*).$$

Therefor we conclude that $(a^*, x^*)$ is a regularized variationally stationary EPEC point. For the other direction, suppose $(a^*, x^*)$ is a regularized variationally stationary EPEC point. Then $x^* \in \mathrm{SOL}_\mu(a^*)$, and for each $i$, there exists $w_i^*$ satisfying

$$(0, w_i^*) \in \partial_{(a_i,x)} \hat{f}_i^{\mu,\beta}(a^*, x^*), \qquad \sum_{i=1}^{N} w_i^* = 0.$$

Set $z_i^* = (a^*, x^*)$ for all $i$. Then $z^* \in \mathcal{L}$. Define $y_i^* = (0, \dots, 0, \dots, 0, w_i^*)$, where all action blocks are zero and the final block is $w_i^*$. Then $y_i^* \in S_i^{\mu,\beta}(z_i^*)$ for all $i$ so $y^* \in S^{\mu,\beta}(z^*)$. Moreover, $\sum_{i=1}^{N} y_i^* = 0$ because all action blocks are zero and the shared-state blocks satisfy $\sum_i w_i^* = 0$, so $y^* \in \mathcal{L}^{\perp}$. Therefore $(z^*, y^*)$ solves the regularized linkage problem.

### A.8 Proof of Theorem 3.5

Fix an outer iteration $\nu$. By assumption, the regularized linkage problem has a solution and $S^\nu$ has elicitable maximal monotonicity at level $\theta^\nu$, with $\kappa^\nu > \theta^\nu$. Therefore, by the convergence theorem for the progressive decoupling algorithm (Royset and Wets, 2021, 10.23), the inner progressive decoupling iteration converges to a solution of the regularized linkage problem (15). Hence the pair returned by the inner loop satisfies. Now let $(z^*, y^*)$ be a cluster point of $\{(z^\nu, y^\nu)\}$. Passing to a subsequence without relabeling, suppose $z^\nu \to z^*$ and $y^\nu \to y^*$. Since $z^\nu \in \mathcal{L}$ for every $\nu$, and $\mathcal{L}$ is a closed subspace, we have $z^* \in \mathcal{L}$. Similarly, we have $y^* \in \mathcal{L}^\perp$. It remains to show that $y^* \in S(z^*)$. Since $y^\nu \in S^\nu(z^\nu)$, we have $(z^\nu, y^\nu) \in \operatorname{gph} S^\nu$. Because $(z^\nu, y^\nu) \to (z^*, y^*)$ and $\operatorname{LimOut}_{\nu\to\infty} \operatorname{gph} S^\nu \subset \operatorname{gph} S$ it follows that $(z^*, y^*) \in \operatorname{gph} S$., which inturn implies that $y^* \in S(z^*)$. So $(z^*, y^*)$ solves the unregularized linkage problem. By proposition 3.4, there exists a common pair $(a^*, x^*)$ such that $z_i^* = (a^*, x^*)$ for all $i = 1, \ldots, N$. Therefore $(a^*, x^*)$ is a variationally stationary Nash point with respect to $\hat{f}_i$.