Recasting the Aiyagari Model of Income into a Mean Field Game Problem

1 Introduction

In this paper, we seek to replicate the work done Achdou, Han, Lasry, Lions, and Moll in their article "Income and Wealth Distribution in Macroeconomics: A Continuous-Time Approach". The paper focuses on casting the Aiyagari Income Model as a Mean Field Game problem, that can be solved in terms of the associated Hamilton-Jacobi Bellman (HJB) and Focker-Plank (FP) equations. In this paper we review the Aiyagari model and the derivation of the MFG system.

For the purposes of this paper we add a small adjustment to the simulation in regards to the total labor supply. As currently formulated total labor supply remains constant throughout the simulation, however we elect to incorporate the idiosyncratic unemployment statuses of households in order to more accurately approximate how the system responds to shocks in labor supply. The motivation behind this adjustment comes from research done by Professor Anthony Klotz at Texas A&M who coined the term "The Great Resignation". The term describes the sudden jump in resignations amidst the COVID 19 pandemic. This phenomenon emphasizes the importance of considering labor supply changes when modeling household-firm interactions. Its important to note that the Aiyagari Model of Income does incorporate idiosyncratic state changes in employment, however its traditionally left out of the total labor supply computation. Throughout this paper we present a brief primer of the Aiyagari model of income that's been modified to incorporate idiosyncratic labor supply, along with necessary background of the HJB and KFP equations, before finally introducing a derivation of the Mean Field Game representation of our labor adjusted income model.

2 Aiyagari Model of Income: Introduction

The Aiyagari Model, is a macroeconomic framework that models the interaction between households and firms. Specifically, it models how households save and spend portions of their wealth in the face of idiosyncratic income shocks brought on by employers, and incomplete market information. The model assumes that households are heterogeneous in that they experience different income levels and wealth. The idiosyncratic income shocks are also a key feature of the model, since it describes unpredictable events in the lifetime of a household such as illness, job loss, or other personal economic changes like medical emergencies.

Under the Aiyagari framework, households consume in order to maximize the utility of their consumption. On the other hand households save not only to smooth consumption over their lifetime but also as a precaution against future income shocks. Our variation of the Aiyagari model will also feature a representative firm, that hires labor and purchases capital. The firm is also subject to the idiosyncratic productivity shocks brought on by sporadic changes in labor productivity. An equilibrium solution represents a distribution of wealth such that households are optimally choose their savings and spending to optimize for utility and smooth consumption.

Now we move on to the mathematical formulation of the Aiyagari model in terms of dynamic stochastic differential equations.

3 Aiyagari Model of Income: Formalization

3.1 Households

In order to define the necessary mathematical formulations of this problem we first define a few terms from the household's perspective, which will be used throughout this paper.

$$z_t = \text{Household labor output} \tag{1}$$
$$w_t = \text{Household wages} \tag{2}$$
$$r_t = \text{Risk-free interest rate} \tag{3}$$
$$a_t = \text{Household wealth} \tag{4}$$
$$c_t = \text{Household spending/consumption} \tag{5}$$
$$D = \text{Borrowing limit} \tag{6}$$

3.1.1 Utility Functional:

As stated above, households spend portions of their wealth to gain some sort of utility from their consumption, whether that utility be in the form of satisfaction, happiness, or value. Their goal is often to smoothen consumption and maximize utility. This goal is given in the form of the following utility functional in continuous time.

$$U = \max_{\{c_t\}} E_{t_0} \int_{t_0}^{\infty} e^{-\rho t} u(c_t) dt \tag{7}$$

ρ in this case is traditionally known as the rate of time preference. It reflects the household's preferences for present consumption over future consumption. In other words a $\rho \approx 0$ means that that a household prioritizes current spending over future spending since it increases the present utility. Whereas a higher ρ will discount the present value of the utility gained through consumption, and will place a higher weight on future consumption.

In discrete time analogue of the household utility function can be expressed as

$$U = \max_{c_t} \sum_t^{\infty} e^{-\rho t} u(c_t) \tag{8}$$

where ρ, again, is the degree to which a household prioritizes current consumption verses future consumption. If $\rho \approx 1$ then the household prioritizes future spending over current spending.

3.1.2 Utility Function:

The utility function $u(c_t)$ in equation (6) and (7) quantifies the satisfaction, happiness, or value an individual received from consumption at time t. the utility function is typically non-decreasing, since higher consumption will often lead to higher utility. However in order to capture the diminishing marginal utility of consumption, we impose that our utility function is strictly concave. In other words as consumption increase the marginal utility gained from consuming one additional unit decreases. Given those key features we can propose a few useful utility function. The first of which is called constant relative risk aversion (CRRA).

$$u(c) = \frac{c^{1-\gamma}}{1-\gamma} \tag{9}$$

Or we can use exponential utility.

$$u(c) = \frac{-1}{\theta} e^{-\theta c} \tag{10}$$

And lastly we can consider logarithmic utility

$$u(c) = \ln(c) \tag{11}$$

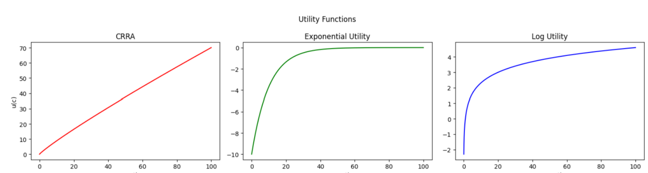

Figure 1: Utility Functions

3.1.3 Household Dynamics:

Households must also adhere to certain spending dynamics that relate their consumption wealth and income.

$$da_t = (z_t w_t + r_t a_t - c_t)dt \tag{12}$$

From the above equation we can observe that the change in wealth is function of a households wage income, income via risk-free investment, and the current consumption. The function

$$s(t, a, z) = z_t w_t + r_t a_t - c_t \tag{13}$$

will represent the amount saved by a household at time t, and will be important when solving the system using a mean field game framework.

We also suject household consumption to a borrowing constraint, which expressly prohibits a household from taking on an excessive amount of debt D.

$$a_t \geq -D \tag{14}$$

We also impose the following state constraint boundary condition, to ensure that constraint (14) is never violated.

$$c_t \leq z_t w_t + r_t(1 + a_t) - a_{t+1} \tag{15}$$

3.2 Firms:

On the other side of the interaction with households will be a representative Neoclassical firm. In order to define the necessary mathematical formulations of the firm-side problem we will again define a few terms from the firm's perspective. These terms in addition to those defined for Households will be used bellow.

$$A_t = \text{Total factor production} \tag{16}$$
$$\alpha = \text{Proportion of investment allocated to capital} \tag{17}$$
$$K_t = \text{Total capital supply} \tag{18}$$
$$L = \text{Total labor supply} \tag{19}$$
$$\delta = \text{Depreciation rate of capital} \tag{20}$$
$$Y_t = \text{Firm's output/productivity} \tag{21}$$
$$\tag{22}$$

A firms productivity is fully characterised by a firms investment in capital and labor. This is a popular production function that was first developed by Charles Cobb and Paul Douglas, and its the same production function used by the Aiyagari income model.

$$Y_t = AK_t^{\alpha} L_t^{1-\alpha} \tag{23}$$

3.2.1 Total Factor Production:

In the simplest case of the Aiyagari model A, sometimes refered to as the total factor productivity (TFP), is constant. However A_t can also be modeled as a function of time to reflect technological progress or other factors affecting productivity. A simple time-dependant model of the TFP can be represented as an exponential where A_0 is the initial level of productivity and γ^* is the rate of technological progress. In this way we model TFP in way that is analogous to Moore's law which describes the exponential growth of the number of transistors on a microchip. A phenomenon which has a direct effect on the productivity of digital systems.

$$A_t = A_0 e^{\gamma^* t} \tag{24}$$

3.2.2 Total Labor Supply:

Once again in the simplest case of the Aiyagari model the total labor supply L is considered constant. However L_t can also be modeled as a function of time to reflect growth in population or changes in labor supply. We will formalize this adjustment bellow in section 4.

3.2.3 Firm Side Problem & Derived Quantities:

Regardless of how K_t or L_t are expressed the firm's aim to maximize it production output remains the same. The problem can be given in terms of the productivity of the firm, appreciation/depreciation of assets, and wage spending.

$$\max_{K,L}\{AK_t^{\alpha}N^{1-\alpha} - (r_t + \delta)K_t - w_t L_t\} \tag{25}$$

Our aim will be to study the household behavior instead of the firm-side behavior, so we will not explicitly be using (25). However the firm's first order conditions with respect to capital supply K_t allows us to derive an explicit expression for the interest rate r_t.

$$r_t = \alpha A K_t^{\alpha-1} L_t^{1-\alpha} - \delta \tag{26}$$

And the firm's first order condition with respect to labor supply L allows to nail down an explicit expression for wages w_t

$$w_t = (1-\alpha)AK^{\alpha}L_t^{-\alpha} \tag{27}$$

3.2.4 Idiosyncratic Income Shocks:

In reality households are subject to idiosyncratic income shocks. In continuous time we can either represent these income shock as a diffusion process, where the change in income (z_t) can be expressed as

$$dz = \mu(z)dt + \sigma^2 dB_t \tag{28}$$

where sigma is the standard deviation of the magnitude of the shocks and $\mu(z)$ is drift of income. There are two ways of approaching the construction of $\mu(z)$. First we can assume that income slowly drifts up as a result of raises in wage or inflation adjustment. In this case

$$\mu(z) = mt + b \tag{29}$$

where m is the rate at which income drifts up and b is the average income at time 0. This construction may be useful if the reader seeks to implement the model for sufficiently large time horizon. However in the short term, we can treat z_t as a mean reverting process. Under this assumption, we can construct $\mu(z)$ as an Ornstein-Uhlenbeck Process.

$$\mu(z) = \theta(\bar{z} - z) \tag{30}$$

Where θ is the rate of mean reversion, and $\bar{z}$ is the average income to which z_t reverts.

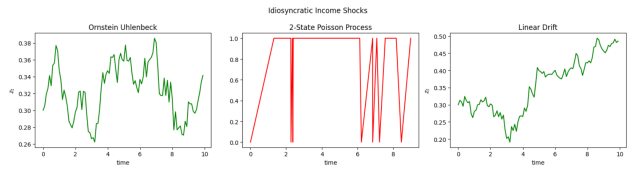

Figure 2: Idiosyncratic Income Shocks

4 Labor Supply Adjusted Model

Incorporating a change in employment or unemployment status into the total labor supply term in the Aiyagari Model, which traditionally uses a more simplified labor market representation, involves modifying the model to account for dynamic changes in labor force participation based on individual states which are also subject to stochastic changes. To implement this adjustment we adopt Achdou, Han, Lasry, Lions, and Mol's use of a 2 state Poisson process to model income idiosyncrasy.

A two-state Poisson jump process is a type of stochastic process that alternates between two distinct states, 0 and 1, with transitions occurring randomly at certain rates. We can denote the two states as $S_0 =$ unemployed and $S_1 =$ employed, the transitions between these states are governed by Poisson processes with different intensity rates.

Let λ_{01} be the rate at which transitions occur from S_0 to S_1 and λ_{10} be the rate for transitions from S_1 to S_0. The process can be described by the following transition probabilities:

$$\mathbb{P}(S_{t+\Delta t} = 1 | S_t = 0) = \lambda_{01}\Delta t \tag{31}$$

$$\mathbb{P}(S_{t+\Delta t} = 0 | S_t = 1) = \lambda_{10}\Delta t \tag{32}$$

However we propose that λ_{01} and λ_{10} are also subject to random fluctuations. We argue that the transition rates must not be fixed and instead ought to be sensitive to macroeconomic fluctuations. For instance, during a recession, job-separation rates might increase (due to layoffs and business closures), and job-finding rates might decrease (due to reduced hiring). Conversely, in a booming economy, job-finding rates might increase, and job-separation rates might decrease. Furthermore changes in government policy, such as increased unemployment benefits or job creation programs, can affect the incentives for individuals to find or leave jobs. Similarly, external shocks like technological changes or global events can rapidly alter the labor market landscape. Given these realistic conditions we assume that our transition rates are uniformly distributed.

The job-finding rate λ_{01} is drawn from a uniform distribution as follows:

$$\lambda_{01} \sim \text{Uniform}(a_{01}, b_{01}) \tag{33}$$

where a_{01} and b_{01} are the lower and upper bounds of the distribution, respectively. Similarly, the job-separation rate λ_{10} is drawn from a uniform distribution:

$$\lambda_{10} \sim \text{Uniform}(a_{10}, b_{10}) \tag{34}$$

where a_{10} and b_{10} are the lower and upper bounds of the distribution for λ_{10}, respectively.

We then use the above formulation of idiosyncratic employment changes to dynamically simulate the total labor supply at each time step in our simulation. We do this by letting L_t be the dynamically adjusting average described bellow.

$$L_t = \frac{z_0\lambda_{01} + z_1\lambda_{10}}{\lambda_{01} + \lambda_{10}} \tag{35}$$

This formula computes a weighted average of the productivities (z_j) for employed and unemployed households, where the weights are the transition probabilities between the states. This can be interpreted as an expectation of income or productivity level, taking into account the likelihood of being in either state. This allows L_t to reasonably estimate the total labor supply/productivity at any given time step.

Luckily, our labor supply adjustment does not greatly impact the deviating of our mean field game representation of the Aiyagari model, So we proceed by presenting a general derivation of the two core PDEs of a mean field game, before recasting the Labor Ajusted Aiyagari model.

5 General MFG System Derivation

5.1 General HJB Equation Derivation

In this section we derive the Hamilton-Jacobi-Bellman equation's generally.The Hamilton-Jacobi-Bellman (HJB) equation is a cornerstone in the theory of optimal control. It provides a necessary condition for optimality across a wide range of control problems, including both deterministic and stochastic systems. The derivation below outlines the fundamental steps involved in arriving at the HJB equation.

Assume a control system described by the differential equation

$$\dot{x}(t) = f(x(t), u(t), t) \tag{36}$$

where $x(t)$ represents the system state at time t, $u(t)$ denotes the control input, and f is a function defining the system dynamics. The objective is to find a control policy $u(t)$ that minimizes the cost functional

$$J = g(x(T)) + \int_{t_0}^{T} L(x(t), u(t), t)\, dt \tag{37}$$

where $g(x(T))$ is the terminal cost, and $L(x(t), u(t), t)$ is the running cost.

Step 1: Value Function. Define the value function V as the minimum cost-to-go (see principle of least action) from a state x at time t, under the given system dynamics and cost structure:

$$V(x(t),t) = \min_u \left\{ g(x(T)) + \int_t^T L(x(\tau), u(\tau), \tau)\, d\tau \right\} \tag{38}$$

Step 2: Principle of Optimality. The principle of optimality states that any optimal path's subpath is also optimal for its subproblem. This allows expressing the value function at $t + dt$ as:

$$\begin{aligned} V(x(t+dt), t+dt) = \min_u \{ & L(x(t), u(t), t)dt \\ & + V(x(t) + \dot{x}(t)dt, t+dt) \} \end{aligned} \tag{39}$$

Step 3: Hamilton-Jacobi-Bellman Equation. By expanding $V(x(t) + \dot{x}(t)dt, t+dt)$ using Taylor's series and neglecting higher-order terms, we obtain:

$$\begin{aligned} V(x(t),t) + \frac{\partial V}{\partial t} dt + \nabla V \cdot \dot{x}(t) dt \\ = \min_u \{ L(x(t), u(t), t)dt + V(x(t), t) \} \end{aligned} \tag{40}$$

After simplifying, the HJB equation emerges as:

$$\frac{\partial V}{\partial t} + \min_u \{ L(x,u,t) + \nabla V \cdot f(x,u,t) \} = 0 \tag{41}$$

This partial differential equation is central to solving optimal control problems by determining the value function $V(x,t)$ and thereby the optimal control policy.

5.2 General Focker Plank Equation

In this section we derive the general Kologmorov Focker-Plank equation. To do this We consider a continuous-time stochastic process described by the Stochastic Differential Equation (SDE)

$$dX_t = \mu(X_t, t)dt + \sigma(X_t, t)dW_t$$

where X_t is the state variable at time t, $\mu(X_t, t)$ is the drift coefficient, $\sigma(X_t, t)$ is the diffusion coefficient, and dW_t represents the increment of a Wiener process (or Brownian motion).

Step 1: Chapman-Kolmogorov Equation. The Chapman-Kolmogorov equation for two times $s < t$ states the probability density function $p(x,t|y,s)$, of the process being in state x at time t, given it was in state y at time s, satisfies

$$p(x,t|z,0) = \int p(x,t|y,s)p(y,s|z,0)dy.$$

Step 2: Infinitesimal Generator. The infinitesimal generator L, acting on functions f of the process' state, is given by

$$Lf(x) = \lim_{\Delta t \to 0} \frac{\mathbb{E}[f(X_{t+\Delta t})|X_t = x] - f(x)}{\Delta t}.$$

For our SDE, this is

$$Lf(x) = \mu(x,t)\frac{\partial f}{\partial x} + \frac{1}{2}\sigma^2(x,t)\frac{\partial^2 f}{\partial x^2}.$$

Step 3: Derivation of the Fokker-Planck Equation. Considering a small time increment Δt and using the properties of X_t, the Fokker-Planck equation is derived as

$$\frac{\partial p}{\partial t} = -\frac{\partial}{\partial x}[\mu(x,t)p] + \frac{1}{2}\frac{\partial^2}{\partial x^2}[\sigma^2(x,t)p].$$

This equation describes how the probability density of the state variable X_t evolves over time due to both deterministic drifts and stochastic diffusions.

5.3 Aiyagari Model HJB Equation

In this section we will recast the Aiyagari model, where the idiosyncratic income changes are represented by a 2 state Poisson Process, into an HJB equation. To do this we will cook a bellman equation that serves the encodes similar features to the Aiyagari model above.

Consider the following income fluctuation problem in discrete time. Periods are of length Δ, individuals discount the future with discount factor

$$\beta(\Delta) = e^{-\rho\Delta} \tag{42}$$

and individuals with income y_j keep their income with probability

$$p_j(\Delta) = e^{-\lambda_j\Delta} \tag{43}$$

and switch to state y_{-j} with probability $1 - p_j(\Delta)$. The Bellman equation for this problem is:

$$\begin{aligned} v_j(a_t) = \max_c \{ & u(c)\Delta + \beta(\Delta)\left(p_j(\Delta)v_j(a_{t+\Delta})\right. \\ & \left. + (1 - p_j(\Delta))v_{-j}(a_{t+\Delta})\right)\} \end{aligned} \tag{44}$$

subject to

$$a_{t+\Delta} = \Delta y_j + ra_t - c + a_t \tag{45}$$

$$a_{t+\Delta} \geq a \tag{46}$$

functionally the Bellman equation encodes a households desire to maximize current utility while balancing the probability of future income shocks on the capacity for consumption to produce utility.

For $j = 1,2$. We will momentarily take $\Delta \to 0$ so we can use that for Δ small

$$\begin{aligned} \beta(\Delta) &= e^{-\rho\Delta} \\ &\approx 1 - \rho\Delta \end{aligned} \tag{47}$$

$$\begin{aligned} p_j(\Delta) &= e^{-\lambda_j\Delta} \\ &\approx 1 - \lambda_j\Delta. \end{aligned} \tag{48}$$

Substituting these into (4) 4we have

$$\begin{aligned} v_j(a_t) = \max_c \{ & u(c)\Delta + (1 - \rho\Delta)\left((1 - \lambda_j\Delta)v_j(a_{t+\Delta})\right. \\ & \left. + \lambda_j\Delta v_{-j}(a_{t+\Delta})\right)\} \end{aligned} \tag{49}$$

subject to (45) and (46). Subtracting $(1-\rho\Delta)v_j(a)$ from both sides and rearranging, we get

$$\begin{aligned} \Delta v_j(a_t) = \max_c \{ & u(c)\Delta + (1 - \rho\Delta)\left(v_j(a_{t+\Delta}) - v_j(a_t)\right) \\ & + \lambda_j\Delta\left(v_{-j}(a_{t+\Delta}) - v_j(a_{t+\Delta})\right)\} \end{aligned} \tag{50}$$

Dividing by Δ, taking $\Delta \to 0$ and using that

$$\begin{aligned} & \lim_{\Delta \to 0} \frac{v_j(a_{t+\Delta}) - v_j(a_t)}{\Delta} \\ &= \lim_{\Delta \to 0} \frac{v_j(\Delta y_j + ra_t - c + a_t) - v_j(a_t)}{\Delta} \\ &= v_j'(a_t)(y_j + ra_t - c) \end{aligned} \tag{51}$$

we find (52); The HJB equation for the Aiyagari model.

$$\begin{aligned} \rho v_j(a) = \max_c \{ & u(c) + v_j'(a)(y_j + ra - c) \\ & + \lambda_j(v_{-j}(a) - v_j(a))\}, \end{aligned} \tag{52}$$

5.4 Aiyagari Model Focker Plank Equation

Recall that under our labor adjustment a household can either be employed or unemployed, and that the transition from one state to another follows a two-state Poisson process. For the sake of stability we assign an unemployed household a low productivity state and vice versa; $z_j \in \{z^L, z^H\}$. Also recall that wealth evolves as follows.

$$a_t = a_{t+\Delta} - \Delta s_j(a_t)).$$

We let $G_j(a,t)$ define the wealth distribution CDF of our population of households. Such that the bellow quantity represents the fraction of people with income z_j and wealth below a.

$$G_j(a,t) = \mathbb{P}(a_t \leq a, z_t = z_j)$$

We claim that the probability of an individual having wealth bellow a is the following. Intuitively this makes sense since the probability of a household having wealth bellow a at $t+\Delta$ is equal to the probability that a household has wealth bellow a at the previous time step plus the probability that the household's current wealth is greater then a at time t but has the capacity to cross bellow the threshold a during the next time step.

$$\mathbb{P}(a_{t+\Delta} \leq a) = \underbrace{\mathbb{P}(a_t \leq a)}_{\text{already below threshold } a} + \underbrace{\mathbb{P}(a \leq a_t \leq a - \Delta s_j(a))}_{\text{cross threshold } a} \tag{53}$$

which is equivalent to

$$\mathbb{P}(a_{t+\Delta} \leq a) = \mathbb{P}(a_t \leq a - \Delta s_j(a)) \tag{54}$$

Now in order to compute the joint probability of $\mathbb{P}(a_t \leq a, z_t = z_j)$ we incorporate the probabilities of income shocks as defined in (47) and (48).

$$\mathbb{P}(a_{t+\Delta} \leq a, y_{t+\Delta} = y_j) = (1-\Delta_j)\Pr(a_t \leq a - \Delta s_j(a), y_t = y_j) + \Delta_{-j}\Pr(a_t \leq a - \Delta s_{-j}(a), y_t = y_{-j}). \tag{55}$$

We then use the definition of G_j, which yields the following.

$$G_j(a, t+\Delta) = (1-\Delta_j)G_j(a - \Delta s_j(a), t) + \Delta_{-j}G_{-j}(a - \Delta s_{-j}(a), t) \tag{56}$$

We then divide out by Δ, then take $\Delta - \to 0$. Using the fact that $g(a) = \partial_a G(a)$, we get the KFP equation.

$$0 = -\frac{d}{da}[s_j(a)g(a)] - \lambda_j g(a) + \lambda_{-j} g_{-j}(a),$$

References

Achdou, Y., Han, J., Lasry, J.-M., Lions, P.-L., & Moll, B. (2021). Income and wealth distribution in macroeconomics: A continuous-time approach. *The Review of Economic Studies*, *89*(1), 45-86. Retrieved from https://doi.org/10.1093/restud/rdab002 doi: 10.1093/restud/rdab002

Andersson, A. (n.d.). *Introduction to the hamilton-jacobi-bellman equation.* Online. Retrieved from https://www.math.chalmers.se/ donnerda/StochasticControl.

Dustmann, C., Schönberg, U., & Stuhler, J. (2016). Labor supply shocks, native wages, and the adjustment of local employment*. *The Quarterly Journal of Economics*, *132*(1), 435-483. Retrieved from https://doi.org/10.1093/qje/qjw032 doi: 10.1093/qje/qjw032

Klotz, A. (2022). The great resignation is still here, but whether it stays is up to leaders. In *Oecd forum.*

Risken, H. (1996). *The fokker-planck equation methods of solution and applications*. Berlin: Springer.