Simulating quantum circuits with non-Clifford noise

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Background

What is a quantum computer?

Input State

$$|\psi\rangle = \alpha\,|0\rangle + \beta\,|1\rangle$$

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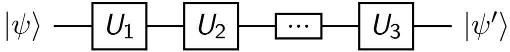

The problem with classical simulations of quantum circuits

$$|\psi\rangle = \begin{bmatrix} \alpha_1 \\ \alpha_2 \end{bmatrix}$$

$$|\psi\rangle \otimes |\phi\rangle = \begin{bmatrix} \alpha_1 \\ \vdots \\ \alpha_4 \end{bmatrix}$$

$$|\psi\rangle \otimes |\phi\rangle \otimes |\lambda\rangle = \begin{bmatrix} \alpha_1 \\ \vdots \\ \vdots \\ \alpha_8 \end{bmatrix}$$

- State grows exponentially, 2^n for n qubits.
- The current state of the art allows for the classical simulation of a 54-qubit system(Pednault et al 2019)

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$$|s_i\rangle \in \mathcal{S} := \{ s \mid \exists C_i \in \mathcal{C} \text{ s.t } s = C_i |0\rangle \}$$

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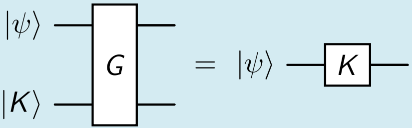
Resource States & Gadgets



s.t $\mathrm{G} \in \mathcal{C}$ & $|K\rangle \in \mathcal{S}$

K-Gadget

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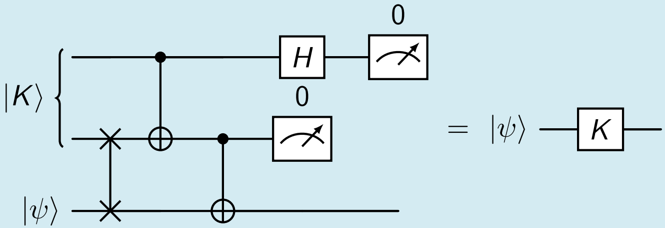

K-Gadget

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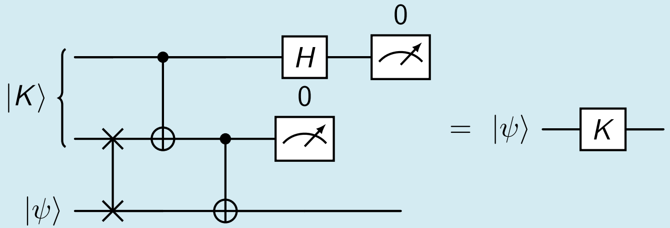
$$K = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \mapsto \begin{pmatrix} a & c & b & d \end{pmatrix}$$



$$|K\rangle = \sum^{4} c_i |s_i\rangle, \quad |s_i\rangle \in \mathcal{S} \implies 4^c \text{ \# of states}$$

Rank 4 K-Gadget

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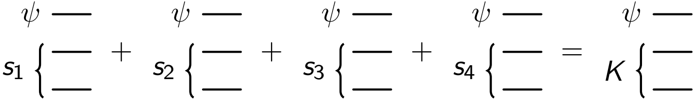

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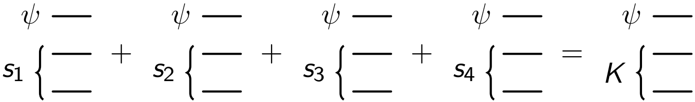
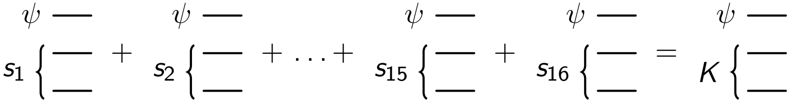
$$|K\rangle = \sum^{4} c_i \, |s_i\rangle$$



$$|K\rangle = \sum^{4^2} c_i \, |s_i\rangle$$



Rank 2 K-Gadget

Theorem - Extent Bound on Kraus Resource State

Given an arbitrary contraction operator $K \in \mathbb{R}^{2\times 2}$, the associated column-wise flattened vector has stabilizer decomposition

$$|K\rangle = \sum_{i}^{r} c_i |s_i\rangle, \quad |s_i\rangle \in \mathcal{S} \text{ such that}$$

$$\sum_{i}^{r} |c_i| \leq 1.268$$

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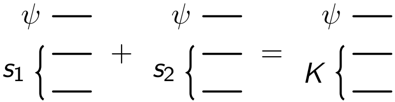
The small extent and numerical simulations indicate that most Kraus operators can be reasonably represented by a **rank 2** stabilizer decomposition!

Rank 2 K-Gadget

$$|K\rangle = \sum^{2} c_i |s_i\rangle$$



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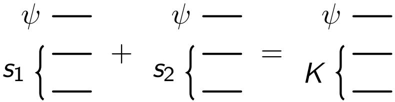

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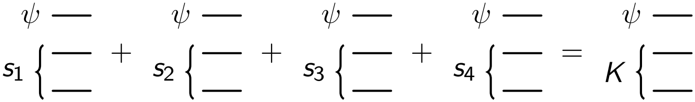

K-gadget Compression

So we choose 2 stabilizer states $|\tilde{0}\rangle$ and $|\tilde{1}\rangle$ to serve as our basis

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$$|K\rangle^{\otimes c} = \left(\frac{|\tilde{0}\rangle + \alpha|\tilde{1}\rangle}{\beta}\right)^{\otimes c} = \frac{1}{\sqrt{K(\mathbb{F}_2^c)}} \sum_{x \in \mathbb{F}_2^c} \alpha^{|x|} |\tilde{x}_1 \tilde{x}_2 \dots \tilde{x}_t\rangle$$

$$K(\mathcal{L}) = \sum_{x,y \in \mathcal{L}} \nu^{|x+y|} \alpha^{|x|+|y|} \text{ , with } \nu = \langle \tilde{0}|\tilde{1}\rangle$$

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$$|K^{\otimes c}\rangle \approx |\mathcal{L}\rangle = \frac{1}{\sqrt{K(\mathcal{L})}} \sum_{x \in \mathcal{L}} \alpha^{|x|}\,|\tilde{x}_1\tilde{x}_2 \ldots \tilde{x}_t\rangle$$

Approximation Error

$$\delta(\mathcal{L}) = 1 - |\langle K^{\otimes c}|\mathcal{L}\rangle|^2$$

$$\langle K^{\otimes c}|\mathcal{L}\rangle = \frac{1}{\sqrt{K(\mathbb{F}_2^1)^c K(\mathcal{L})}} \sum_{y\in\mathcal{L}} \sum_{x\in\mathbb{F}_2^c} \alpha^{|x|+|y|} \nu^{|x+y|}$$

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Outstanding Questions

- How do we choose $|\tilde{0}\rangle$ and $|\tilde{1}\rangle$?

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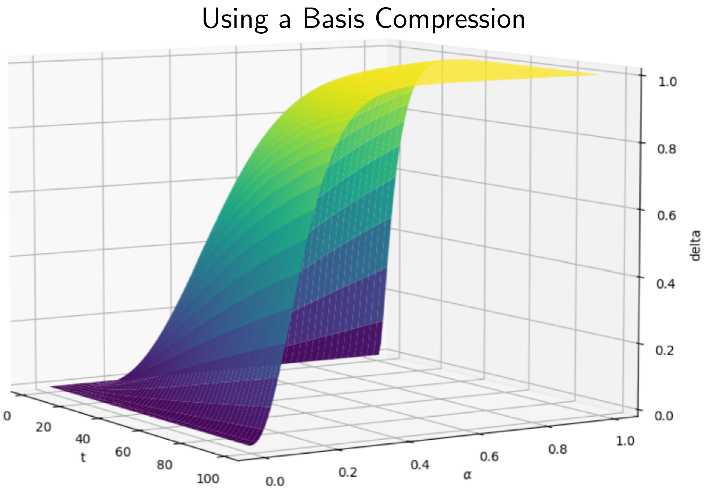
Outstanding Questions

- How do we choose $|\tilde{0}\rangle$ and $|\tilde{1}\rangle$?
  Such that α is minimized
- How do we choose $\mathcal{L}$?
  Such that the elements of $\mathcal{L}$ have small hamming weights

Choose $\mathcal{L}$ to be the basis of $\mathbb{F}_2^c$

$$\mathcal{L} = \mathcal{B}_2^c \cup \vec{0}$$

Approximation Error



Approximation Error

α	$\delta = .0001$	$\delta = .001$	$\delta = .01$
$\alpha = .0001$	37782	225963	2038344
$\alpha = .0005$	3011	12167	85849
$\alpha = .001$	1075	3783	22714
$\alpha = .005$	107	302	1227
$\alpha = .01$	41	108	383
$\alpha = .05$	5	11	31
$\alpha = .1$	2	4	12

Table: Maximum t with $\nu = \frac{1}{\sqrt{2}}$

Summary

1. Map K to a resource state $|K\rangle$
2. Find stabilizer states that minimize α
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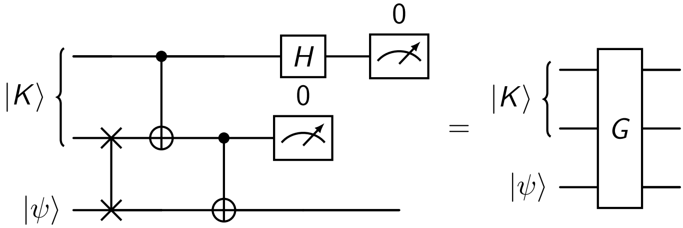

Example - Amplitude Dampening

$$\xi_{AD} = \left\{ K_1 = \begin{bmatrix} 1 & 0 \\ 0 & \sqrt{1-p} \end{bmatrix}, K_2 = \begin{bmatrix} 0 & \sqrt{p} \\ 0 & 0 \end{bmatrix} \right\}$$

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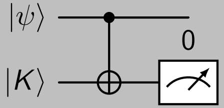
$$K_1 \mapsto |K_1\rangle = \begin{pmatrix} 1 & \sqrt{1-p} \end{pmatrix},$$



Simulations pt 3.

$$|K_1\rangle = \frac{|+\rangle + \alpha\,|0\rangle}{\beta}, \quad \alpha = \frac{1-\sqrt{1-p}}{\sqrt{2(1-p)}}, \quad \beta = \sqrt{2(1-p)}$$

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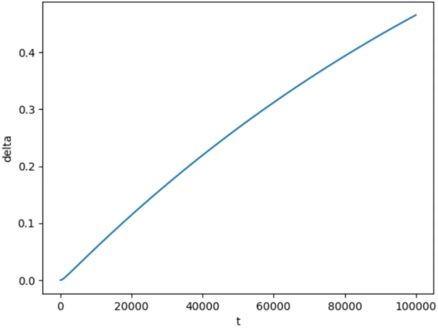
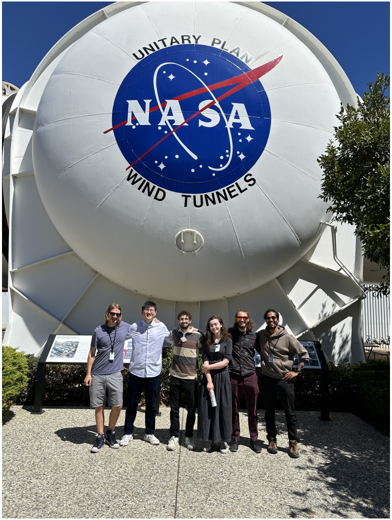


Figure: $p = .01$

α	$\delta = .0001$	$\delta = .001$	$\delta = .01$
$\alpha = .003562$	173	505	2215

We can simulate 2215 applications of amplitude dampening within 99% fidelity using only 1.4GB of memory!

Acknowledgements



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2. NASA Ames Research Center for allowing us to visit their facility in Mountain View, CA.

Questions!